1. The curve C has equation

$$y = 31\sinh x - 2\sinh 2x \qquad x \in \mathbb{R}$$

Determine, in terms of natural logarithms, the exact x coordinates of the stationary points of C.

(7)

Solution 1

$$y = 31\sinh x - 2\sinh 2x$$

$$\Rightarrow \frac{dy}{dx} = 31\cosh x - 2(2)\cosh 2x$$

$$\Rightarrow \frac{dy}{dx} = 31\cosh x - 4\cosh 2x = 31\cosh x - 4(2\cosh^2 x - 1)$$

$$\frac{dy}{dx} = 0 \Rightarrow 31\cosh x - 4(2\cosh^2 x - 1) = 0$$

$$\Rightarrow 31c - 8c^2 + 4 = 0$$

$$\Rightarrow 8c^2 - 31c - 4 = 0$$

$$\Rightarrow (8c+1)(c-4) = 0$$

$$\Rightarrow \cosh x = 4 \text{ or } \cosh x = -1/8$$

From formula booklet,

$$\cosh x = \alpha \Rightarrow x = \ln(\alpha \pm \sqrt{\alpha^2 - 1})$$

$$\Rightarrow x = \ln(4 \pm \sqrt{4^2 - 1}) = \ln(4 \pm \sqrt{15})$$

2. In an Argand diagram, the points A and B are represented by the complex numbers $-3 + 2\text{i}$ and $5 - 4\text{i}$ respectively. The points A and B are the end points of a diameter of a circle C.

(a) Find the equation of C, giving your answer in the form

$$|z - a| = b \qquad a \in \mathbb{C},\ b \in \mathbb{R}$$

(3)

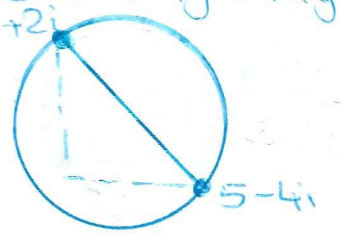
The circle D, with equation $|z - 2 - 3\text{i}| = 2$, intersects C at the points representing the complex numbers z_1 and z_2

(b) Find the complex numbers z_1 and z_2

(6)

Solution 2a

Visualising may help



Find diameter by using Pythagoras:

$A = (-3+2i)$, $B = (5-4i)$

$$\text{Diameter}^2 = (5-(-3))^2 + (-4-2)^2$$
$$= 8^2 + (-6)^2$$
$$= 100$$

$\Rightarrow$ Diameter $= 10$ $\Rightarrow$ radius $= 5$

$$\text{Centre} = \frac{1}{2}(A+B) = \frac{1}{2}\left[(-3+2i)+(5-4i)\right]$$
$$= \frac{1}{2}\left[2-2i\right] = 1 - i$$

So equation is

$$|z-(1-i)| = 5$$
$$\Rightarrow |z-1+i| = 5$$

Solution 2b

Now $|x+iy-1+i|=5$

and $|x+iy-2-3i|=2$

$\Rightarrow (x-1)^2+(y+1)^2=5^2$

$(x-2)^2+(y-3)^2=2^2$

$\Rightarrow x^2-2x+1+y^2+2y+1=25$ (*)

$x^2-4x+4+y^2-6y+9=4$ (**)

(*) - (**)

$\Rightarrow 2x-3+8y-8=21$

$\Rightarrow 2x+8y=32 \Rightarrow \boxed{x+4y=16}$ (***)

Substitute in (*) to solve for y

(*) $\Rightarrow \boxed{x}^2-2\boxed{x}+y^2+2y=23$

$\Rightarrow \boxed{(16-4y)}^2-2\boxed{(16-4y)}+y^2+2y=23$

$\Rightarrow 16^2-2\times4\times16y+16y^2-32+8y+y^2+2y=23$

$\Rightarrow 256-128y+16y^2-32+8y+y^2+2y=23$

$\Rightarrow 17y^2-118y+201=0$

$\Rightarrow \boxed{y=\frac{67}{17}}$ or $\boxed{y=3}$

Substitute in (***) $\boxed{x}+4\boxed{y}=16$ to solve for x:

$x=16-4\boxed{\left(\frac{67}{17}\right)}=\boxed{\frac{4}{17}}$ or $x+4\boxed{(3)}=16 \Rightarrow \boxed{x=4}$

So $z_1=\boxed{\frac{4}{17}}+\boxed{\frac{67}{17}}i$ or $z_2=\boxed{4}+\boxed{3}i$

3. A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination.
The concentration of antibodies, x, measured in micrograms (μg) per millilitre (ml) of blood, is modelled by the differential equation

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$$

where t is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

(4)

Initially,

- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 μg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

(8)

A second dose of the vaccine has to be given to try to ensure that it is fully effective. It is only safe to give the second dose if the concentration of antibodies in the bloodstream of the patient is less than 5 μg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

(2)

Solution 3a

Auxiliary Equation $100m^2 + 60m + 13 = 0$

$$\Rightarrow m = \frac{-60 \pm \sqrt{60^2 - 4(100)(13)}}{200}$$

$$\Rightarrow m = \frac{-60 \pm \sqrt{1600}\, i}{200} = \frac{-60 \pm 40i}{200}$$

$$\Rightarrow m = -\frac{3}{10} \pm \frac{1}{5}i \Rightarrow m = \boxed{-0.3} \pm \boxed{0.2}i$$

$$\Rightarrow x = \boxed{e^{\boxed{-0.3t}}(A\cos\boxed{0.2}t + B\sin\boxed{0.2}t)}$$

PI : $\boxed{x = 2}$ (this is found using $x = k \Rightarrow \frac{dx}{dt} = 0 \Rightarrow \frac{d^2x}{dt^2} = 0$)
(then equating LHS & RHS of eqn.)
(So $13 \times k = 26 \Rightarrow k = 2$)

So $x = \boxed{e^{-0.3t}(A\cos 0.2t + B\sin 0.2t)} + \boxed{2}$

Solution 3b

From part a,

$$x = e^{-0.3t}(\boxed{A}\cos 0.2t + \boxed{B}\sin 0.2t) + 2$$

Also, $t=0$, $x=0$ $\Rightarrow$ $0 = e^0(A+0)+2$

$\Rightarrow \boxed{A = -2}$

$\Rightarrow x = e^{-0.3t}(\boxed{-2}\cos 0.2t + B\sin 0.2t) + 2$

product rule

$$\Rightarrow \boxed{\frac{dx}{dt}} = \boxed{-0.3e^{-0.3t}(-2\cos 0.2t + B\sin 0.2t) + e^{-0.3t}(0.4\sin 0.2t + 0.2B\cos 0.2t)}$$

Also, $t=0$, $\boxed{\frac{dx}{dt} = 10}$

$\Rightarrow \boxed{10} = \boxed{-0.3(-2+0) + (0+0.2B)}$

$\Rightarrow \boxed{10} = \boxed{0.6 + 0.2B}$

$\Rightarrow \frac{9.4}{0.2} = B \Rightarrow B = 47$

$$\Rightarrow x = e^{-0.3t}(\boxed{47}\sin 0.2t \boxed{-2}\cos 0.2t) + 2 \qquad (*)$$

Solve for $\frac{dx}{dt} = 0$

Tidying up

$$\boxed{\frac{dx}{dt}} = \boxed{-0.3e^{-0.3t}(-2\cos 0.2t + 47\sin 0.2t) + e^{-0.3t}(0.4\sin 0.2t + 0.94\cos 0.2t)}$$

$= e^{-0.3t}(0.6\cos 0.2t - 14.1\sin 0.2t + 0.4\sin 0.2t + 9.4\cos 0.2t)$

$= e^{-0.3t}((0.6+9.4)\cos 0.2t + (-14.1+0.4)\sin 0.2t)$

$= e^{-0.3t}(10\cos 0.2t - 13.7\sin 0.2t) \qquad (**)$

When $\frac{dx}{dt} = 0$ in $(**)$, we have $10\cos 0.2t - 13.7\sin 0.2t = 0$

$\Rightarrow \frac{\sin 0.2t}{\cos 0.2t} = \frac{100}{137} \Rightarrow \tan 0.2t = 0.729927\ldots$

use radians in calculator

$\Rightarrow 0.2t = 0.630\ldots \Rightarrow t = 3.15$ weeks

Substitute in $(*)$: $x = e^{-0.3\times 3.15}(47\sin 0.2\times 3.15 - 2\cos 0.2\times 3.15) + 2$

$\Rightarrow x = 12.1\ \mu g/ml$

Solution 3c

Substitute $t=10$ in $(*)$:

$$x = e^{-0.3\times 10}\left(47\sin(0.2\times 10) - 2\cos(0.2\times 10)\right) + 2$$

$$\Rightarrow x = 4.16986\ldots \ \mu g/ml$$

Since $x = 4.19\ldots < 5$, then we have that the model suggests that it would be safe to give the second dose.

4. (a) Use de Moivre's theorem to prove that

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

(5)

(b) Hence find the distinct roots of the equation

$$1 + 7x - 56x^3 + 112x^5 - 64x^7 = 0$$

(5)

giving your answer to 3 decimal places where appropriate.

Solution 4a

Since 7θ (and particularly $\sin 7\theta$) is being used, and de Moivre's theorem is required, use the expansion of $(\cos\theta + i\sin\theta)^7$:

$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$. Let $c = \cos\theta$, $s = \sin\theta$

$= (c+is)^7 = c^7 + 7c^6 is + \frac{7!}{5!2!}c^5 i^2 s^2 + \frac{7!}{4!3!}c^4 i^3 s^3 + \frac{7!}{3!4!}c^3 i^4 s^4 + \frac{7!}{2!5!}c^2 i^5 s^5 + 7c i^6 s^6 + i^7 s^7$

$= c^7 + 7ic^6 s - 21c^5 s^2 - 35ic^4 s^3 + 35c^3 s^4 + 21ic^2 s^5 - 7cs^6 - is^7$

Equate imaginary parts:

$\sin 7\theta = 7\boxed{c^6}s - 35\boxed{c^4}s^3 + 21\boxed{c^2}s^5 - s^7$

since expression we are trying to prove has only $\sin\theta$, convert all $\cos\theta$s to $\sin\theta$s. So use formula: $\cos^2\theta \equiv 1 - \sin^2\theta$

$= 7\boxed{(c^2)^3}s - 35\boxed{(c^2)^2}s^3 + 21\boxed{c^2}s^5 - s^7$

$= \boxed{7(1-s^2)^3 s} - \boxed{35(1-s^2)^2 s^3} + \boxed{21(1-s^2)s^5} - s^7$

$= \boxed{7(1-3s^2+3s^4-s^6)s} - \boxed{35(1-2s^2+s^4)s^3} + \boxed{21s^5 - 21s^7} - s^7$

$= \boxed{7s - 21s^3 + 21s^5 + 7s^7} \boxed{- 35s^3 + 70s^5 - 35s^7} + \boxed{21s^5 - 21s^7} - s^7$

Tidying up

$= 7s - (35+21)s^3 + (21+70+21)s^5 + (-7-35-21-1)s^7$

$= 7s - 56s^3 + 112s^5 - 64s^7$

$= 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

Solution 4b

From part a,

(*) $\sin 7\theta + 1 = 1 + 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

Let $x = \sin\theta$ and solve $0 = 1 + 7x - 56x^3 + 112x^5 - 64x^7$

(*) $\Rightarrow \sin 7\theta + 1 = 0$

$\Rightarrow \sin 7\theta = -1$

$\Rightarrow 7\theta = -450°, -90°, 270°, 630°, \ldots$

$\Rightarrow \theta = \frac{-450°}{7}, -\frac{90°}{7}, \frac{270°}{7}, \frac{630°}{7}, \ldots$

$\Rightarrow \sin\theta = -0.901, -0.223, 0.623, 1$

5. (a)

$$y = \tan^{-1} x$$

Assuming the derivative of $\tan x$, prove that

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

(3)

$$f(x) = x\tan^{-1} 4x$$

(b) Show that

$$\int f(x)\,dx = Ax^2 \tan^{-1} 4x + Bx + C\tan^{-1} 4x + k$$

where k is an arbitrary constant and A, B and C are constants to be determined.

(5)

(c) Hence find, in exact form, the mean value of f(x) over the interval $\left[0, \frac{\sqrt{3}}{4}\right]$

(2)

Solution 5a

$y = \tan^{-1} x$

Since question is asking to assume derivative of $\tan x$, then realise that tan is required (not $\tan^{-1}$)

$\Rightarrow x = \tan y$ (*) tan of both sides

$\Rightarrow \frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$ (**)

(*) $\Rightarrow x^2 = \tan^2 y$

But $\sec^2 y \equiv 1 + \tan^2 y$

So (**) $\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

Solution 5b

Let $I = \int \boxed{x}\,\boxed{\tan^{-1} 4x}\,dx$

Let $\boxed{u = \tan^{-1} 4x}$, $\boxed{\frac{dv}{dx} = x}$

$\Rightarrow \boxed{\frac{du}{dx} = \frac{4}{1+(4x)^2}}$, $\boxed{v = \frac{x^2}{2}}$

Integrating by parts gives:

$$I = \boxed{\tfrac{1}{2}x^2}\,\boxed{\tan^{-1} 4x} - \int \boxed{\tfrac{1}{2}x^2} \times \boxed{\frac{4}{1+(4x)^2}}\,dx$$

$$= \frac{1}{2}x^2 \tan^{-1} 4x - \int \frac{2x^2}{1+16x^2}\,dx$$

If numerator has a power greater than or equal to power in denominator, do long divison or spot a trick!

$$= \frac{1}{2}x^2 \tan^{-1} 4x - \frac{1}{8}\int \frac{16x^2}{1+16x^2}\,dx$$

$$= \frac{1}{2}x^2 \tan^{-1} 4x - \frac{1}{8}\int \frac{1+16x^2-1}{1+16x^2}\,dx$$

$$= \frac{1}{2}x^2 \tan^{-1} 4x - \frac{1}{8}\int 1 - \frac{1}{1+16x^2}\,dx$$

$$= \frac{1}{2}x^2 \tan^{-1} 4x - \frac{1}{8}\int 1 - \frac{1}{1+(4x)^2}\,dx$$

$$= \frac{1}{2}x^2 \tan^{-1} 4x - \frac{1}{8}\left(x - \frac{1}{4}\tan^{-1} 4x\right) + k$$

$$= \frac{1}{2}x^2 \tan^{-1} 4x - \frac{1}{8}x + \frac{1}{32}\tan^{-1} 4x + k$$

So $A = \frac{1}{2}$, $B = -\frac{1}{8}$, $C = \frac{1}{32}$

6.

$$\mathbf{M} = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Given that $k \neq 4$, find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(4)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect.

$$\begin{aligned} 2x + 5y + 7z &= 1 \\ x + y + z &= p \\ 2x + y - z &= 2 \end{aligned}$$

(3)

(c) (i) Find the value of q for which the following planes intersect in a straight line.

$$\begin{aligned} 4x + 5y + 7z &= 1 \\ x + y + z &= q \\ 2x + y - z &= 2 \end{aligned}$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)

Solution 6a

$$\det M = k\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 7\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= k(-1-1) - 5(-1-2) + 7(1-2)$$

$$= -2k + 15 - 7 = 8 - 2k = 2(4-k)$$

$$M^T = \begin{pmatrix} k & 1 & 2 \\ 5 & 1 & 1 \\ 7 & 1 & -1 \end{pmatrix}$$

$$\text{Cofactors} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 7 & -1 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 7 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} k & 2 \\ 7 & -1 \end{vmatrix} & -\begin{vmatrix} k & 1 \\ 7 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} k & 2 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 5 & 1 \end{vmatrix} \end{pmatrix}$$

$$\text{Cofactors} = \begin{pmatrix} -2 & 12 & -2 \\ 3 & -(k+14) & 7-k \\ -3 & 10-k & k-5 \end{pmatrix}$$

$$\Rightarrow M^{-1} = \frac{1}{8-2k}\begin{pmatrix} -2 & 12 & -2 \\ 3 & -(k+14) & 7-k \\ -3 & 10-k & k-5 \end{pmatrix}$$

Solution 6b

$$2x + 5y + 7z = 1$$
$$x + y + z = p$$
$$2x + y - z = 2$$

k=2 in part a

$$\Rightarrow \begin{pmatrix} 2 & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$$\Rightarrow M^{-1} = \frac{1}{8 - 2(2)} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -(2+14) & 7-2 \\ -1 & 8 & -3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -16 & 5 \\ -1 & 8 & -3 \end{pmatrix}$$

$$\Rightarrow M^{-1}M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -16 & 5 \\ -1 & 8 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2 + 12p - 4 \\ 3 - 16p + 10 \\ -1 + 8p - 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 12p - 6 \\ 13 - 16p \\ 8p - 7 \end{pmatrix}$$

Co-ordinates are $\left(3p - \frac{3}{2}, \frac{13}{4} - 4p, 2p - \frac{7}{4}\right)$

Solution 6ci and 6cii

Label equations:

$$4x+5y+7z=1 \quad (1)$$
$$x+y+z=q \quad (2)$$
$$2x+y-z=2 \quad (3)$$

(3) $\Rightarrow z=2x+y-2$

Substitute $z=2x+y-2$ in (1):

$$4x+5y+7(2x+y-2)=1$$
$$\Rightarrow 4x+5y+14x+7y-14=1$$
$$\Rightarrow 18x+12y=15$$
$$\Rightarrow 6x+4y=5 \quad (4)$$

Substitute $z=2x+y-2$ in (2):

$$x+y+(2x+y-2)=q$$
$$\Rightarrow 3x+2y-2y=q$$
$$\Rightarrow 3x+2y=q+2$$

x 2 to match (4)

$$\Rightarrow 6x+4y=2q+4 \quad (5)$$

Equations (4) and (5) equated give:

$$6x+4y=5=2q+4$$

$$\Rightarrow q=\frac{1}{2}$$

7.

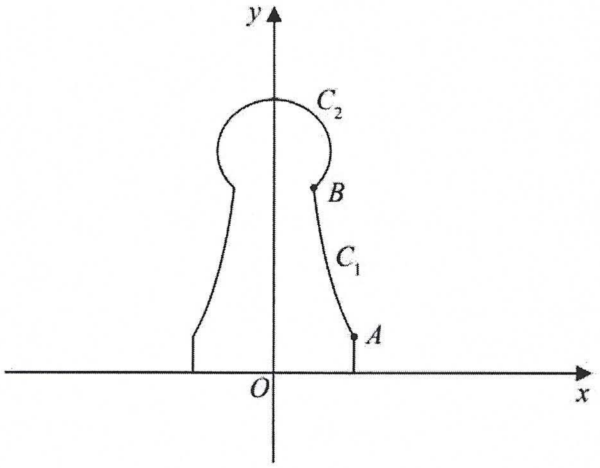


Figure 1

A student wants to make plastic chess pieces using a 3D printer. Figure 1 shows the central vertical cross-section of the student's design for one chess piece. The plastic chess piece is formed by rotating the region bounded by the y-axis, the x-axis, the line with equation $x = 1$, the curve C_1 and the curve C_2 through 360° about the y-axis.

The point A has coordinates (1, 0.5) and the point B has coordinates (0.5, 2.5) where the units are centimetres.

The curve C_1 is modelled by the equation

$$x = \frac{a}{y + b} \qquad 0.5 \leqslant y \leqslant 2.5$$

(a) Determine the value of a and the value of b according to the model.

(2)

The curve C_2 is modelled to be an arc of the circle with centre (0, 3).

(b) Use calculus to determine the volume of plastic required to make the chess piece according to the model.

(9)

Solution 7a

At $A = (1, 0.5)$: $1 = \frac{a}{0.5+b} \Rightarrow a = 0.5 + b$ ①

At $B = (0.5, 2.5)$: $0.5 = \frac{a}{2.5+b} \Rightarrow 0.5(2.5+b) = a$ ②

Equating equations ① and ② gives:

$0.5 + b = 0.5(2.5 + b)$

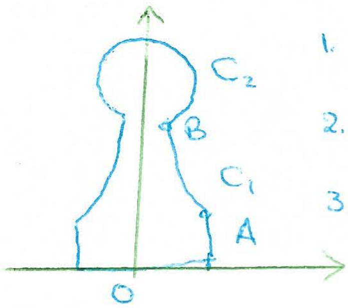
$\Rightarrow 0.5 + b = 1.25 + 0.5b$

$\Rightarrow 0.5b = 0.75 \Rightarrow b = 1.5$

Substituting $b = 1.5$ in ① gives

$a = 0.5 + 1.5 = 2$

Solution 7b



Find volume in 3 parts:

1. For curve C_2
2. For curve C_1
3. For cylinder at the base

For curve C_2, find limits:

$x^2+(y-3)^2=r^2$ ← curve C_2 has centre $(0,3)$

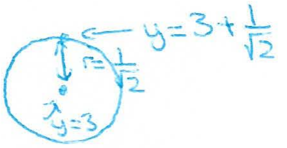
B lies on C_2 at $(0.5, 2.5)$

So $0.5^2+(2.5-3)^2=0.5$

$\Rightarrow r^2=0.5$

$\Rightarrow r=\frac{1}{\sqrt{2}}$

$\Rightarrow$ Highest point is at $y=3+\frac{1}{\sqrt{2}}$



Therefore, $V_2=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}\boxed{x^2}\,dy$

$x^2+(y-3)^2=0.5$ (Curve C_2)

$$=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}\boxed{0.5-(y-3)^2}\,dy$$

$$=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}0.5-(y^2-6y+9)\,dy$$

$$=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}-y^2+6y-8.5\,dy$$

$$=\pi\left[-\frac{1}{3}y^3+3y^2-8.5y\right]_{2.5}^{3+\frac{1}{\sqrt{2}}}$$

$$=\pi\left(\left(-\frac{1}{3}\left(3+\frac{1}{\sqrt{2}}\right)^3+3\left(3+\frac{1}{\sqrt{2}}\right)^2-8.5\left(3+\frac{1}{\sqrt{2}}\right)\right)-\left(\frac{1}{3}\times 2.5^3+3\times 2.5^2-8.5\times 2.5\right)\right)$$

$$=\pi\left(\left(\frac{-45+\sqrt{2}}{6}-\left(\frac{-185}{24}\right)\right)\right)=\pi\left(\frac{5+4\sqrt{2}}{24}\right)$$

Solution 7b continued

For curve C_1, we have limits 0.5 and 2.5 :

$$V_1 = \pi \int_{0.5}^{2.5} x^2 \, dy$$

$x = \dfrac{2}{y+1.5}$ (Curve C_1)

$$= \pi \int_{0.5}^{2.5} \left(\frac{2}{y+1.5} \right)^2 dy$$

$$= \pi \int_{0.5}^{2.5} 4(y+1.5)^{-2} \, dy$$

$$= \pi \left[-4(y+1.5)^{-1} \right]_{0.5}^{2.5}$$

$$= \pi(-4)\left((2.5+1.5)^{-1} - (0.5+1.5)^{-1} \right)$$

$$= -4\pi \left(\frac{1}{4} - \frac{1}{2} \right) = -4\pi \left(-\frac{1}{4} \right)$$

$$= \pi$$

For cylinder at the base :

Radius = $\boxed{1}$, Height = y-co-ordinate of A = $\boxed{0.5}$

$$V_{cylinder} = \pi \boxed{r}^2 \boxed{h} = \pi \times \boxed{1}^2 \times \boxed{0.5}$$

$$= \frac{1}{2}\pi$$

Volume of Chess Piece $= V_1 + V_2 + V_{cylinder}$

$$= \pi + \pi \left(\frac{5+4\sqrt{2}}{24} \right) + \frac{1}{2}\pi$$

$$= \pi \left(\frac{41}{24} + \frac{\sqrt{2}}{6} \right) \text{cm}^3$$