1. A student was asked to answer the following:

For the complex numbers $z_1 = 3 - 3\text{i}$ and $z_2 = \sqrt{3} + \text{i}$, find the value of $\arg\left(\frac{z_1}{z_2}\right)$

The student's attempt is shown below.

Line 1 → $\arg(z_1) = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$

Line 2 → $\arg(z_2) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Line 3 → $\arg\left(\frac{z_1}{z_2}\right) = \frac{\arg(z_1)}{\arg(z_2)}$

Line 4 → $= \left(\frac{\pi}{4}\right) \Big/ \left(\frac{\pi}{6}\right) = \frac{3}{2}$

The student made errors in line 1 and line 3

Correct the error that the student made in

(a) (i) line 1

(ii) line 3

(2)

(b) Write down the correct value of $\arg\left(\frac{z_1}{z_2}\right)$

(1)

Solution 1ai

In line 1, the student should have written

$$\arg(z_1) = \tan^{-1}\left(-\frac{3}{3}\right) = \tan^{-1}(-1) = -\frac{\pi}{4} \quad \left(\text{or } \frac{7\pi}{4}\right)$$

Solution 1aii

In line 3, the student should have written

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Solution 1b

$$\arg\left(\frac{z_1}{z_2}\right) = \boxed{\arg(z_1)} - \boxed{\arg(z_2)}$$

$$= \boxed{\tan^{-1}\left(\frac{-3}{3}\right)} - \boxed{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \boxed{\frac{-\pi}{4}} - \boxed{\frac{\pi}{6}}$$

$$= \boxed{\frac{-3\pi}{12}} - \boxed{\frac{2\pi}{12}}$$

$$= -\frac{5\pi}{12}$$

2. **In this question you must show all stages of your working.**

A college offers only three courses: Construction, Design and Hospitality.

Each student enrols on just one of these courses.

In 2019, there was a total of 1110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

In 2020 the number of students enrolled on

- Construction **increased** by 1.25%
- Design **increased** by 2.5%
- Hospitality **decreased** by 2%

In 2020, the total number of students at the college increased by 0.27% to 2 significant figures.

(a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019.

(ii) Using your variables from part (a)(i), write down **three** equations that model this situation.

(4)

(b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019.

(4)

Solution 2ai

Let C = number of Construction students

D = number of Design students

H = number of Hospitality students

Solution 2aii

Increase in number of students in 2020:

$$1110 \times 0.0027 = 3$$

Total no. of students in 2019: $C + D + H = 1110$

In 2019, 370 more on C than H : $C = H + 370$

C ↑ 1.25%, D ↑ 2.5%, H ↓ 2% : $1.0125C + 1.025D + 0.98H = 1113$

Solution 2b

Equations from 2a are:

$$C + D + H = 1110$$
$$C \quad - H = 370$$
$$1.0125C + 1.025D + 0.98H = 1113$$

Converting to matrix equation gives:

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}}_{M} \begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}$$

$$\det M = 1(0 \times 0.98 - (-1)(1.025)) - 1(1 \times 0.98 - (-1)(1.0125)) + 1.025$$
$$= 1.025 - 0.98 - 1.0125 + 1.025$$
$$= 0.0575$$

$$M^T = \begin{pmatrix} 1 & 1 & 1.0125 \\ 1 & 0 & 1.025 \\ 1 & -1 & 0.98 \end{pmatrix}$$

$$\Rightarrow \operatorname{Adj} M = \begin{pmatrix} \begin{vmatrix} 0 & 1.025 \\ -1 & 0.98 \end{vmatrix} & -\begin{vmatrix} 1 & 1.025 \\ 1 & 0.98 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1.0125 \\ -1 & 0.98 \end{vmatrix} & \begin{vmatrix} 1 & 1.0125 \\ 1 & 0.98 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1.0125 \\ 0 & 1.025 \end{vmatrix} & -\begin{vmatrix} 1 & 1.0125 \\ 1 & 1.025 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$$

$$\Rightarrow \operatorname{Adj} M = \begin{pmatrix} 1.025 & 0.045 & -1 \\ -1.9925 & -0.0325 & 2 \\ 1.025 & -0.0125 & -1 \end{pmatrix}$$

Solution 2b continued

$$M^{-1} = \frac{1}{\det M} \operatorname{Adj} M$$

$$= \begin{pmatrix} 17.826 & 0.783\ldots & -17.391 \\ -34.652\ldots & -0.565\ldots & 34.783 \\ 17.826 & -0.217\ldots & -17.391 \end{pmatrix}$$

Since

$$\begin{pmatrix} C \\ D \\ H \end{pmatrix} = M^{-1} \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 720 \\ 40 \\ 350 \end{pmatrix}$$

In 2020
$\Rightarrow$ 720 studied construction
40 " " Design
350 " " Hospitality

3. $$\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \qquad \text{where } a \text{ is a constant}$$

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$$

(6)

Triangle T has vertices A, B and C.

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$ where $n \in \mathbb{N}$

Given that

- triangle T has an area of $5\,\text{cm}^2$
- triangle T' has an area of $1215\,\text{cm}^2$
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$

(b) determine

(i) the value of n

(ii) the value of a

(5)

Solution 3a

When $n=1$, $LHS = M^1 = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$

$RHS = \begin{pmatrix} 3^1 & \frac{a}{2}(3^1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$

Hence LHS = RHS

So true for $n=1$.

Suppose statement true for $n=k$.

$$M^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$$

Prove true for $n=k+1$.

Now

$$M^{k+1} = M^k \cdot M = M^k \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & a3^k + \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & \frac{a}{2}(3^{k+1}-1) \\ 0 & 1 \end{pmatrix}$$

Hence statement true for $n=k+1$

4. (i) Given that

$$z_1 = 6e^{\frac{\pi}{3}i} \text{ and } z_2 = 6\sqrt{3}e^{\frac{5\pi}{6}i}$$

show that

$$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$$

(3)

(ii) Given that

$$\arg(z-5) = \frac{2\pi}{3}$$

determine the least value of $|z|$ as z varies.

(3)

Solution 4i

$\boxed{z_1} = 6e^{\pi/3 i} = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \boxed{3+3\sqrt{3}i}$

$\boxed{z_2} = 6\sqrt{3}e^{5\pi/6 i} = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 6\sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \boxed{-9+3\sqrt{3}i}$

$\Rightarrow \boxed{z_1+z_2} = \boxed{(3+3\sqrt{3}i)} + \boxed{(-9+3\sqrt{3}i)} = \boxed{-6} + \boxed{6\sqrt{3}i}$

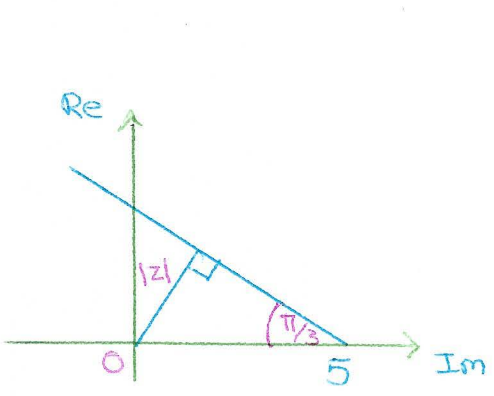
$\Rightarrow |z_1+z_2| = \sqrt{\boxed{(-6)^2} + \boxed{(6\sqrt{3})^2}} = \sqrt{36+108} = \sqrt{144} = \boxed{12}$

Also $\arg(z_1+z_2) = \tan^{-1}\left(\frac{\boxed{6\sqrt{3}}}{\boxed{-6}}\right) = \tan^{-1}(-\sqrt{3}) = \boxed{\frac{2\pi}{3}}$

Hence $z_1+z_2 = \boxed{12}e^{\boxed{\frac{2\pi}{3}}i}$

Solution 4ii

$$\arg(z-5) = \frac{2\pi}{3}$$



From the diagram,

$\sin\left(\frac{\pi}{3}\right) = \frac{|z|}{5}$ (sin = opp/hyp)

$\Rightarrow |z| = 5\sin\left(\frac{\pi}{3}\right) = \frac{5\sqrt{3}}{2}$

Solution 3bi

$T' = M^n T$

Area T' $= (\det M^n)$ Area T

$\Rightarrow \quad 1215 = (3^n) \times 5$

$\Rightarrow \quad 3^n = 243$

$\Rightarrow \quad n = 5$

Vertex $A(2,-2)$ is transformed to $A'(123,-2)$

$$\Rightarrow \quad \begin{pmatrix} 123 \\ -2 \end{pmatrix} = \begin{pmatrix} 3^5 & \frac{a}{2}(3^5-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \quad 123 = 3^5 \times 2 + (-2)\left(\frac{a}{2}\right)(3^5-1)$$

$$\Rightarrow \quad 123 - 3^5 \times 2 = a(1-3^5)$$

$$\Rightarrow \quad a = 1.5$$

5. (a) Given that

$$y = \arcsin x \qquad -1 \leqslant x \leqslant 1$$

show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

(3)

(b) $$f(x) = \arcsin(e^x) \qquad x \leqslant 0$$

Prove that f(x) has no stationary points.

(3)

Solution 5a

$$y = \arcsin x \Rightarrow x = \sin y$$

$$\Rightarrow \frac{dx}{dy} = \cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

But $x = \sin y \Rightarrow x^2 = \sin^2 y \Rightarrow x^2 = 1 - \cos^2 y$

$$\Rightarrow \cos^2 y = 1 - x^2$$

$$\Rightarrow \cos y = \sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Solution 5b

$f(x) = \arcsin(e^x) \quad (x \leqslant 0)$ by part a and chainrule $\Rightarrow f'(x) = \frac{1}{\sqrt{1-e^{2x}}} \times e^x$

Since $e^x \neq 0$, there are no stationary points as $f'(x) \neq 0$.

6. The cubic equation

$$4x^3 + px^2 - 14x + q = 0$$

where p and q are real positive constants, has roots α, β and γ

Given that $\boxed{\alpha^2 + \beta^2 + \gamma^2 = 16}$

(a) show that $p = 12$

(3)

Given that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{14}{3}$

(b) determine the value of q

(3)

Without solving the cubic equation,

(c) determine the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(4)

Solution 6a

Now $4x^3 + px^2 - 14x + q = 0$

Recall: If $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ, then $\sum\alpha = -\frac{b}{a}$, $\sum\alpha\beta = \frac{c}{a}$, $\alpha\beta\gamma = \frac{-d}{a}$

$$\Rightarrow \boxed{\sum\alpha = -\frac{p}{4}}, \boxed{\sum\alpha\beta = -\frac{14}{4}}, \alpha\beta\gamma = \frac{-q}{4}$$

So

$$\boxed{(\alpha+\beta+\gamma)}^2 = \alpha^2 + \alpha\beta + \alpha\gamma + \beta^2 + \beta\gamma + \beta\alpha + \gamma^2 + \gamma\alpha + \gamma\beta$$

$\alpha^2 + \beta^2 + \gamma^2 = 16$ (from question)

$$= \boxed{\sum\alpha^2} + 2\boxed{\sum\alpha\beta}$$

$$\Rightarrow \boxed{\left(\frac{-p}{4}\right)^2} = \boxed{16} + 2\boxed{\left(-\frac{14}{4}\right)}$$

$$\Rightarrow \boxed{\frac{p^2}{4^2}} = \boxed{16}\boxed{-7} = 9$$

$$\Rightarrow p^2 = 9 \times 16 = 144$$

$$\Rightarrow p = 12$$

Solution 6b

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{14}{3} \Rightarrow \frac{\boxed{\beta\gamma+\alpha\gamma+\alpha\beta}}{\alpha\beta\gamma}=\frac{14}{3}$$

from calculation in part a

$$\Rightarrow \frac{\boxed{-7/2}}{-9/4}=\frac{14}{3}$$

$$\Rightarrow \frac{7}{2}\times\frac{4}{q}=\frac{14}{3}$$

$$\Rightarrow \frac{7}{2}\times 4\times\frac{3}{14}=q$$

$$\Rightarrow q=3$$

Solution 6c

expanding

$$(\alpha- \quad (\beta-1)(\gamma-1)=(\alpha\beta-\alpha-\beta+1)(\gamma-1)$$

$$=\alpha\beta\gamma-\alpha\beta-\alpha\gamma+\alpha-\beta\gamma+\beta+\gamma-1$$

$$=\boxed{\alpha\beta\gamma}-\boxed{(\alpha\beta+\alpha\gamma+\beta\gamma)}+\boxed{(\alpha+\beta+\gamma)}-1$$

part a   part a

$$=\boxed{-\frac{q}{4}}-\boxed{\left(-\frac{7}{2}\right)}\boxed{-\frac{p}{4}}-1$$

part b   part a

$$=\boxed{\frac{-3}{4}}+\frac{7}{2}\boxed{-\frac{12}{4}}-1$$

$$=\frac{-1}{4}-1=-\frac{5}{4}$$

7.

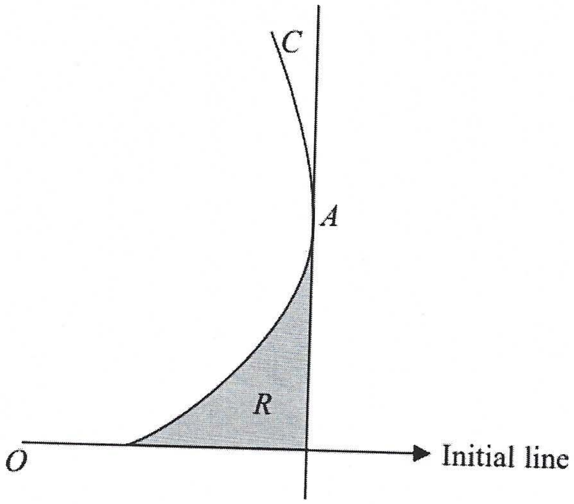


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$r = 1 + \tan\theta \qquad 0 \leqslant \theta < \frac{\pi}{3}$$

Figure 1 also shows the tangent to C at the point A.
This tangent is perpendicular to the initial line.

(a) Use differentiation to prove that the polar coordinates of A are $\left(2, \frac{\pi}{4}\right)$

(4)

The finite region R, shown shaded in Figure 1, is bounded by C, the tangent at A and the initial line.

(b) Use calculus to show that the exact area of R is $\frac{1}{2}(1 - \ln 2)$

(6)

Solution 7a

polar co-ordinates $x = \boxed{\boxed{r}\cos\theta}$, $r = \boxed{1+\tan\theta}$ from question

$\Rightarrow x = \boxed{(1+\tan\theta)}\boxed{\cos\theta} = \cos\theta + \frac{\sin\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}$

multiplying out

$\Rightarrow x = \cos\theta + \sin\theta$

$\Rightarrow \frac{dx}{d\theta} = -\sin\theta + \cos\theta$

$\frac{dx}{d\theta} = 0 \Rightarrow -\sin\theta + \cos\theta = 0 \Rightarrow \tan\theta = 1$

$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$

$\Rightarrow r = 1 + \tan\theta = 1 + 1 = \boxed{2}$

$\Rightarrow$ polar co-ordinates of A (r, θ) are $\left(\boxed{2}, \boxed{\frac{\pi}{4}}\right)$

Solution 7b

Area bounded by curve $= \int \frac{1}{2} r^2 \, d\theta$

Area bdd by curve $= \frac{1}{2}\int_0^{\pi/4} \underbrace{(1+\tan\theta)}_{r}{}^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi/4} (1+2\tan\theta + \boxed{\tan^2\theta})\, d\theta$ expanding brackets

$= \frac{1}{2}\int_0^{\pi/4} (1+2\tan\theta + \boxed{\sec^2\theta - 1})\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4} (2\tan\theta + \sec^2\theta)\, d\theta$ tidying up

$= \int_0^{\pi/4} \left(\boxed{\tan\theta} + \boxed{\frac{1}{2}\sec^2\theta}\right) d\theta$

$= \left[\boxed{-\ln|\cos\theta|} + \boxed{\frac{1}{2}\tan\theta}\right]_0^{\pi/4}$

$= \left(-\ln\cos\frac{\pi}{4} + \frac{1}{2}\tan\frac{\pi}{4}\right) - \left(-\ln\cos 0 + \frac{1}{2}\tan 0\right)$ putting in limits

$= -\ln\frac{1}{\sqrt{2}} + \frac{1}{2} + \underbrace{\ln 1}_{\leftarrow 0} = \frac{1}{2} - \ln\frac{1}{\sqrt{2}} = \frac{1}{2} + \ln\left(\frac{1}{\sqrt{2}}\right)^{-1}$

$= \frac{1}{2} + \ln\sqrt{2}$

polar co-ordinates

Area of $\Delta = \frac{1}{2}xy = \frac{1}{2}\left(2\cos\frac{\pi}{4}\right)\left(2\sin\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2}\sqrt{2} = 1$

So required area = Area of Δ − Area bdd by curve

$= 1 - \left(\frac{1}{2} + \ln\sqrt{2}\right)$

$= \frac{1}{2} - \ln\sqrt{2} = \frac{1}{2} - \ln 2^{1/2}$

$= \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$

8. Two birds are flying towards their nest, which is in a tree.

Relative to a fixed origin, the flight path of each bird is modelled by a straight line.

In the model, the equation for the flight path of the first bird is

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$$

and the equation for the flight path of the second bird is

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

where λ and μ are scalar parameters and a is a constant.

In the model, the angle between the birds' flight paths is 120°

(a) Determine the value of a.

(4)

(b) Verify that, according to the model, there is a common point on the flight paths of the two birds and find the coordinates of this common point.

(5)

The position of the nest is modelled as being at this common point.

The tree containing the nest is in a park.

The ground level of the park is modelled by the plane with equation

$$2x - 3y + z = 2$$

(c) Hence determine the shortest distance from the nest to the ground level of the park.

(3)

(d) By considering the model, comment on whether your answer to part (c) is reliable, giving a reason for your answer.

(1)

Solution 8a

Recall $\underline{a} \cdot \underline{b} = |a||b|\cos\theta$

$$\frac{\begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{2^2 + a^2 + 0^2}\,\sqrt{0^2 + 1^2 + (-1)^2}} = \cos 120°$$

$$\Rightarrow \frac{0 + a - 0}{\sqrt{4+a^2}\,\sqrt{2}} = -\frac{1}{2} \Rightarrow a = -\left(\frac{\sqrt{2}}{2}\right)\sqrt{4+a^2}$$

Squaring both sides gives $a^2 = \frac{2}{4}(4 + a^2) \Rightarrow a^2 = 4$

$\Rightarrow a = -2$ (not +2 sub in

Solution 8b

If there is a common point on flight paths, then

$$\begin{pmatrix}-1\\5\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\0\end{pmatrix}=\begin{pmatrix}4\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned}-1+2\lambda &= 4\\ 5-2\lambda &= -1+\mu\\ 2 &= 3-\mu\end{aligned}$$

$$\Rightarrow \quad \lambda = \frac{5}{2} \quad \text{and} \quad \mu = 1$$

Check LHS and RHS are same for these values of λ and μ.

$$\text{LHS} = \begin{pmatrix}-1\\5\\2\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-2\\0\end{pmatrix}=\begin{pmatrix}-1+5\\5-5\\2\end{pmatrix}=\begin{pmatrix}4\\0\\2\end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix}4\\-1\\3\end{pmatrix}+1\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}4\\0\\2\end{pmatrix}$$

So common point is $\begin{pmatrix}4\\0\\2\end{pmatrix}$

Solution 8c

Perpendicular distance from plane to origin:

$2x-3y+z=2 \quad |n| = \sqrt{2^2+(-3)^2+1^2} = \sqrt{14}$

$\Rightarrow$ shortest distance $= \dfrac{2}{\sqrt{14}}$

Perp. dist from plane containing pt of int to origin:

$2x-3y+z = \begin{pmatrix}4\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix} = 10 \Rightarrow$ shortest distance $= \dfrac{10}{\sqrt{14}}$

$$\text{Min distance} = \frac{10}{\sqrt{14}} - \frac{2}{\sqrt{14}} = \frac{8}{\sqrt{14}}$$

Solution 8d: Model not reliable as nest modelled as a point.

9. $$y = \cosh^n x \qquad n \geqslant 5$$

(a) (i) Show that

$$\frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2} x$$

(4)

(ii) Determine an expression for $\frac{d^4y}{dx^4}$

(2)

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form.

(2)

Solution 9ai

$y = \cosh^n x$ Let $u = \cosh x$

$\Rightarrow y = u^n$, $\frac{dy}{du} = nu^{n-1}$ and $\frac{du}{dx} = \sinh x$

Since $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, $\frac{dy}{dx} = nu^{n-1} \times \sinh x$

So $\frac{dy}{dx} = n\cosh^{n-1} x \sinh x$ $(*)$

Now use product rule to differentiate again:

Let $u = n\cosh^{n-1} x$ $\quad v = \sinh x$

$\Rightarrow \frac{du}{dx} = n\left[(n-1)\cosh^{n-2} x \sinh x\right]$ $\quad \frac{dv}{dx} = \cosh x$

product rule

$\Rightarrow \frac{d^2y}{dx^2} = (\sinh x)\underbrace{\left(n(n-1)\cosh^{n-2} x \sinh x\right)}_{\text{applying } (*) \text{ with different } n} + \left(n\cosh^{n-1} x\right)(\cosh x)$

$= n(n-1)c^{n-2}s^2 + nc^n$

$= n(n-1)c^{n-2}(c^2-1) + nc^n$ $\quad \sinh^2 x = \cosh^2 x - 1$

$= n(n-1)(c^n - c^{n-2}) + nc^n$

$= n(n-1)c^n - n(n-1)c^{n-2} + nc^n$

$= n^2c^n - nc^n - n(n-1)c^{n-2} + nc^n$

$= n^2c^n - n(n-1)c^{n-2}$

$= n^2\cosh^n x - n(n-1)\cosh^{n-2} x$

Solution 9aii

From 9ai,

$$\frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2}x$$

(*) $$\Rightarrow \frac{d^3y}{dx^3} = n^2(nc^{n-1}s) - n(n-1)(n-2)c^{n-3}s$$

$$= n^3c^{n-1}s - n(n-1)(n-2)c^{n-3}s$$

$$= n^2\boxed{(nc^{n-1}s)} - n(n-1)\boxed{(n-2)c^{n-3}s}$$

From part a,

$$\frac{dy}{dx} = nc^{n-1}s$$

$$\Rightarrow \frac{d^2y}{dx^2} = n^2c^n - n(n-1)c^{n-2}$$

$$\Rightarrow \frac{d^4y}{dx^4} = n^2\boxed{(n^2c^n - n(n-1)c^{n-2})} - n(n-1)\left(\boxed{(n-2)^2c^{n-2} - (n-2)(n-3)c^{n-4}}\right)$$

$$= n^4c^n - n^3(n-1)c^{n-2} - n(n-1)(n-2)^2c^{n-2} + n(n-1)(n-2)(n-3)c^{n-4}$$

$$= n^4c^n - c^{n-2}(n^3(n-1) + n(n-1)(n-2)^2) + n(n-1)(n-2)(n-3)c^{n-4}$$

$$= n^4c^n - c^{n-2}(n(n-1)(n^2 + (n-2)^2)) + n(n-1)(n-2)(n-3)c^{n-4}$$

$$= n^4c^n - c^{n-2}(n(n-1)(2n^2 - 4n + 4) + n(n-1)(n-2)(n-3)c^{n-4}$$

$$= n^4c^n - 2nc^{n-2}(n-1)(n^2-2n+2) + n(n-1)(n-2)(n-3)c^{n-4}$$

Solution 9b

When $x=0$ $\quad y=1,\ y'=0\quad y''=n^2-n(n-1)=n$

$y'''=0,$

$$y^{(4)} = n^4 - 2n(n-1)(n^2-2n+2) + n(n-1)(n-2)(n-3)$$

$$= n^4 + n(n-1)(-n^2-n+2)$$

$$= n(n^3 + (n-1)(-n^2-n+2))$$

$$= n(n^3 - n^3 - n^2 + 2n + n^2 + n - 2)$$

$$= n(3n-2)$$

Now $$y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \frac{x^4}{4!}y^{(4)}(0)$$

$$= 1 + \frac{n}{2}x^2 + \frac{(3n^2-2n)}{24}x^4 + \ldots$$