786 Further Maths 1 Ref: 9FM0/01
Paper 1: Core Pure Maths 1 Monday 11 May 2020

① $f(z) = 3z^3 + pz^2 + 57z + q$ where p and q are real constants. Given that $3 - 2\sqrt{2}i$ is a root of the equation $f(z) = 0$.

ⓐ Show all the roots of $f(z) = 0$ on a single Argand diagram (7)

ⓑ Find the value of p and the value of q. (3)

Solution 1a

$\alpha = 3 - 2\sqrt{2}i \Rightarrow \beta = 3 + 2\sqrt{2}i$

$$\begin{aligned}(z-\alpha)(z-\beta) &= z^2 - (\alpha+\beta)z + \alpha\beta \\ &= z^2 - 6z + (3-2\sqrt{2}i)(3+2\sqrt{2}i) \\ &= z^2 - 6z + (9 - 4(2)(-1)) \\ &= z^2 - 6z + 17\end{aligned}$$

If γ is the third root, then

$$\begin{aligned}f(z) &= (z-\alpha)(z-\beta)(3z+a) \\ &= (z^2 - 6z + 17)(3z+a) \\ &= 3z^3 - 6z(3z) + 17(3z) + az^2 - 6az + 17a \\ &= 3z^3 - 18z^2 + 51z + az^2 - 6az + 17a \\ &= 3z^3 + \boxed{(a-18)}z^2 + \boxed{(51-6a)}z + \boxed{17a} \\ &= 3z^3 + \boxed{p}z^2 + \boxed{57}z + \boxed{q}\end{aligned}$$

EXPANDING OUT

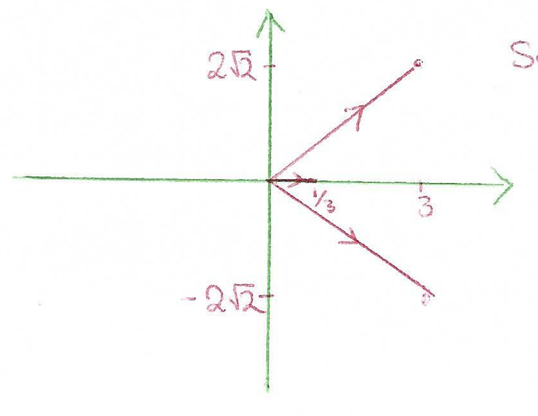
FROM QUESTION

$\Rightarrow 51 - 6a = 57 \Rightarrow a = -1$

So $f(z) = (z-\alpha)(z-\beta)(3z-1)$

$\Rightarrow \gamma$ (the third root) $= 1/3$

$\therefore$ roots are: $\alpha = 3 - 2\sqrt{2}i$, $\beta = 3 + 2\sqrt{2}i$ and $\alpha = 1/3$



Sorry! Not to scale

Solution 1b

From part a, we know:

$$f(z) = 3z^3 + \boxed{(a-18)}z^2 + \boxed{(51-6a)}z + \boxed{17a}$$
$$= 3z^3 + \boxed{p}z^2 + \boxed{57}z + \boxed{q}$$

Also, $a = -1$

$\therefore\ a - 18 = -19 = p$ (comparing co-eff of z^2)

$\therefore\ 17a = q \Rightarrow q = -17$ (comparing constant terms)

$p = -19$ and $q = -17$

② ⓐ Explain why $\int_1^\infty \frac{1}{x(2x+5)}\,dx$ is an improper integral (1)

ⓑ Prove that $\int_1^\infty \frac{1}{x(2x+5)}\,dx = a\ln b$

where a and b are rational numbers to be determined. (6)

Solution 2a

The upper limit of this integral is ∞
$\therefore$ it is improper
(can also say the 'interval is unbounded')

Solution 2b

Convert $\frac{1}{x(2x+5)}$ to partial fractions:

$$\frac{1}{x(2x+5)} \equiv \frac{A}{x} + \frac{B}{2x+5}$$

$$\Rightarrow \quad 1 \equiv A(2x+5) + Bx$$

Let $x=0 \Rightarrow 1 = 5A \Rightarrow A = 1/5$

Let $x = -\frac{5}{2} \Rightarrow 1 = -\frac{5}{2}B \Rightarrow B = -2/5$

Hence $\int_1^\infty \frac{1}{x(2x+5)}\,dx = \int_1^\infty \frac{1}{5x} - \frac{2}{5(2x+5)}\,dx$

$$= \frac{1}{5}\int \frac{1}{x} - \frac{2}{2x+5}\,dx = \frac{1}{5}\Big[\ln x - \ln(2x+5)\Big]_1^\infty$$

$$= \frac{1}{5}\left[\ln\left(\frac{x}{2x+5}\right)\right]_1^\infty = \frac{1}{5}\left[\ln\left(\frac{1}{2+5/x}\right)\right]_1^\infty$$

$$= \frac{1}{5}\left[\lim_{t\to\infty}\ln\left(\frac{1}{2+5/t}\right) - \ln\left(\frac{1}{7}\right)\right] = \frac{1}{5}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{7}\right)\right)$$

$$= \frac{1}{5}\ln\frac{7}{2}$$

$a = \frac{1}{5}$

3

Figure 1

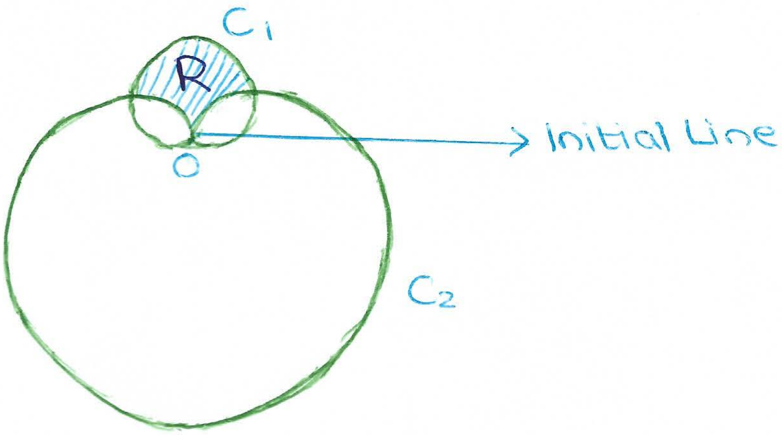


Figure 1 shows a sketch of 2 curves C_1 and C_2 with polar equations:

$$C_1 : r = (1+\sin\theta) \quad 0 \leq \theta < 2\pi$$

$$C_2 : r = 3(1-\sin\theta) \quad 0 \leq \theta < 2\pi$$

The region R lies inside C_1 and outside C_2 and is shown in figure 1.

Show that the area of R is

$$p\sqrt{3} - q\pi$$

where p and q are integers to be determined. (9)

Solution 3

Find intersection points of C_1 and C_2 to determine limits of integral.

$$1+\sin\theta = 3(1-\sin\theta)$$

$$\Rightarrow \quad 4\sin\theta = 2$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Recall: Area $= \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$

Area of R $= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2 \, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 3^2(1-\sin\theta)^2 \, d\theta$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + \sin^2\theta - 9 + 18\sin\theta - 9\sin^2\theta \, d\theta$$

Solution 3 continued

Area of R

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} -8\sin^2\theta + 20\sin\theta - 8 \, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} -4\sin^2\theta + 10\sin\theta - 4 \, d\theta$$

$$= -2\int_{\pi/6}^{5\pi/6} 2\sin^2\theta - 5\sin\theta + 2 \, d\theta$$

$$= -2\int_{\pi/6}^{5\pi/6} 1 - \cos 2\theta - 5\sin\theta + 2 \, d\theta$$

$$= -2\int_{\pi/6}^{5\pi/6} 3 - \cos 2\theta - 5\sin\theta \, d\theta$$

$$= -2\left[3\theta - \frac{1}{2}\sin 2\theta + 5\cos\theta\right]_{\pi/6}^{5\pi/6}$$

$$= -2\left[\left(3\left(\frac{5\pi}{6}\right) - \frac{1}{2}\sin 2\left(\frac{5\pi}{6}\right) + 5\cos\left(\frac{5\pi}{6}\right)\right) - \left(\frac{3\pi}{6} - \frac{1}{2}\sin 2\left(\frac{\pi}{6}\right) + 5\cos\frac{\pi}{6}\right)\right]$$

$$= -2\left(\left(\frac{5}{2}\pi - \frac{9\sqrt{3}}{4}\right) - \left(\frac{9\sqrt{3}}{4} - \frac{\pi}{2}\right)\right)$$

$$= 2\left(\frac{9\sqrt{3}}{2} - 2\pi\right)$$

$$= 9\sqrt{3} - 4\pi$$

4 The plane Π_1 has equation

$$r = 2i + 4j - k + \lambda(i + 2j - 3k) + \mu(-i + 2j + k)$$

where λ and μ are scale parameters

(a) Find a Cartesian equation for Π_1 (4)

The line l has equation

$$\frac{x-1}{5} = \frac{y-3}{-3} = \frac{z+2}{4}$$

(b) Find the co-ordinates of the point of intersection of l with Π_1 (3)

The plane Π_2 has equation

$$\underline{r} \cdot (2i - j + 3k) = 5$$

(c) Find, to the nearest degree, the acute angle between Π_1 and Π_2 (2)

Solution 4a

Vector perpendicular to plane is:

$$\underline{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ -1 & 2 & 1 \end{vmatrix} = \begin{pmatrix} 2-(-6) \\ -(1-3) \\ 2+2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

Equation of plane: position vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

vector perp to plane

$$\Rightarrow 4x + y + 2z = 8 + 4 - 2 = 10$$

$$\Rightarrow 4x + y + 2z = 10$$

Solution 4b (easier alternative available)

Put the equation $4x+y+2z=10$ in terms of x by using $\frac{x-1}{5}=\frac{y-3}{-3}=\frac{z+2}{4}$

$$\frac{x-1}{5}=\frac{y-3}{-3} \Rightarrow y=-3\left(\frac{x-1}{5}\right)+3$$

$$\Rightarrow y=\frac{-3x+3+15}{5}$$

$$\Rightarrow y=\frac{18-3x}{5}$$

$$\frac{x-1}{5}=\frac{z+2}{4} \Rightarrow z=4\left(\frac{x-1}{5}\right)-2$$

$$\Rightarrow z=\frac{4x-4-10}{5}$$

$$\Rightarrow z=\frac{4x-14}{5}$$

Hence $4x+y+2z=10$

$$\Rightarrow 4x+\left(\frac{18-3x}{5}\right)+2\left(\frac{4x-14}{5}\right)=10$$

$$\Rightarrow 20x+18-3x+8x-28=50$$

$$\Rightarrow 25x=60$$

$$\Rightarrow x=\frac{12}{5}$$

$$\Rightarrow y=\frac{18-3\left(\frac{12}{5}\right)}{5}=\frac{90-36}{25}=\frac{54}{25}$$

$$\Rightarrow z=\frac{4\left(\frac{12}{5}\right)-14}{5}=\frac{48-70}{25}$$

$$\Rightarrow z=-\frac{22}{25}$$

Solution 4b

$$\frac{x-1}{5} = \frac{y-3}{-3} = \frac{z+2}{4} \Rightarrow r = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + p\begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$$

Substitute in the equation of the plane found in part (a):

$$4\boxed{x} + \boxed{y} + 2\boxed{z} = 10$$

$$\Rightarrow 4\boxed{(1+5p)} + \boxed{(3-3p)} + 2\boxed{(-2+4p)} = 10$$

$$\Rightarrow 4 + 20p + 3 - 3p - 4 + 8p = 10$$

$$\Rightarrow 25p = 7$$

$$\Rightarrow p = \frac{7}{25}$$

$$\Rightarrow x = 1+5p = 1 + 5 \times \frac{7}{25} = \frac{12}{5}$$

$$y = 3-3p = 3 - 3 \times \frac{7}{25} = \frac{75-21}{25} = \frac{54}{25}$$

$$z = -2+4p = -2 + 4 \times \frac{7}{25} = \frac{-50+28}{25} = \frac{-22}{25}$$

$$\left(\frac{12}{5}, \frac{54}{25}, \frac{-22}{25}\right)$$

Solution 4c

A Vector $\perp$ to Π_1 is : $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

A Vector $\perp$ to Π_2 is : $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Recall : $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$

To find angles between planes:

$$\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}\right| \cos\theta$$

$$\Rightarrow 8 - 1 + 6 = \sqrt{4^2+1^2+2^2}\sqrt{2^2+1^2+(-3)^2} \cos\theta$$

$$\Rightarrow 13 = \sqrt{21}\sqrt{14} \cos\theta$$

$$\Rightarrow \cos\theta = \frac{13}{\sqrt{294}}$$

$$\Rightarrow \theta = 41^\circ \text{ (nearest degree)}$$

5. Two compounds, X and Y, are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations.

$$\frac{dx}{dt} = -5x + 10y - 30$$

$$\frac{dy}{dt} = -2x + 3y - 4$$

(a) Show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$

(b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for x,

(ii) the particular solution for y

A scientist thinks that the chemical reaction will have stopped after 8 mins.

(e) Explain whether this is supported by the model.

Solution 5a

$$\boxed{\frac{dx}{dt} = -5x + 10y - 30}$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10\frac{dy}{dt}}$$

Also, $\boxed{\frac{dy}{dt} = -2x + 3y - 4}$

So $\boxed{\frac{d^2x}{dt^2}} + 2\frac{dx}{dt} + 5x$

$$= \boxed{-5\frac{dx}{dt} + 10\frac{dy}{dt}} + 2\frac{dx}{dt} + 5x$$

$$= -3\boxed{\frac{dx}{dt}} + 10\boxed{\frac{dy}{dt}} + 5x$$

$$= -3\boxed{(-5x + 10y - 30)} + 10\boxed{(-2x + 3y - 4)} + 5x$$

$$= 50$$

Solution 5b

$$\boxed{\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50} \quad (*)$$

$\Rightarrow$ Auxiliary equation is $\boxed{m^2 + 2m + 5 = 0}$

$\Rightarrow m = -1 \pm 2i$

$\Rightarrow \quad x = \boxed{e^{-t}(A\cos 2t + B\sin 2t)}$ $\leftarrow$ CF

Also, when $x = c$, $(*)$ $\Rightarrow$ $5c = 50$ $\Rightarrow \boxed{c = 10}$ $\leftarrow$ PI

Hence (putting together CF and PI) gives:

$$x = \boxed{e^{-t}(A\cos 2t + B\sin 2t)} + \boxed{10} \quad \leftarrow \text{GS}$$

Solution 5c

From the question,

$$\frac{dx}{dt} = -5x + 10y - 30$$

From part b,

$$\boxed{x = e^{-t}(A\cos 2t + B\sin 2t)} + 10 \quad (**)$$

using product rule

$$\Rightarrow \frac{dx}{dt} = e^{-t}\left(\boxed{-2A\sin 2t} + \boxed{2B\cos 2t}\right) - e^{-t}\left(\boxed{A\cos 2t} + \boxed{B\sin 2t}\right)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = e^{-t}\left(\boxed{(2B-A)\cos 2t} - \boxed{(B+2A)\sin 2t}\right)}$$

Substitute in $\boxed{\frac{dx}{dt} = -5x + 10y - 30}$ to give

$$\boxed{e^{-t}\left((2B-A)\cos 2t - (B+2A)\sin 2t\right) = -5e^{-t}(A\cos 2t + B\sin 2t) - 50 + 10y - 30}$$

$$\Rightarrow e^{-t}\left((2B-A+5A)\cos 2t - (B+2A-5B)\sin 2t = 10y - 80\right.$$

$$\left.(2B+4A)\cos 2t + (4B-2A)\sin 2t\right) + 8 \quad (***)$$

$$\Rightarrow y = \frac{e^{-t}}{10}\Big($$

Solution 5di

$t=0, x=2$ $(**)$ $\Rightarrow 2 = e^{0}(A(1) + B(0)) + 10$

$\Rightarrow 2 = A + 10$

$\Rightarrow A = -8$

$t=0, y=5$ $(***)$ $\Rightarrow 5 = \frac{1}{10}((2B-32)(1) + (4B+16)(0)) + 8$

$\Rightarrow 50 - 80 = 2B - 32$

$\Rightarrow B = 1$

$(**) \Rightarrow x = e^{-t}(-8\cos 2t + \sin 2t) + 10$

Solution 5dii

$(***) \Rightarrow y = \frac{e^{-t}}{10}((2 + 4(-8))\cos 2t + (4 - 2(-8))\sin 2t) + 8$

$\Rightarrow y = \frac{e^{-t}}{10}(-30\cos 2t + 20\sin 2t) + 8$

$\Rightarrow y = e^{-t}(-3\cos 2t + 2\sin 2t) + 8$

Solution 5e

For $t > 8$, $x \approx 10$, $y \approx 8$. So x and y remain almost constant which suggests that the chemical reaction has stopped.

⑥ (i) Prove by induction that for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n}(3r+1)(r+2) = n(n+2)(n+3) \qquad (6)$$

(ii) Prove by induction that for all positive <u>odd</u> integers n, $f(n) = 4^n + 5^n + 6^n$ is divisible by 15 (6)

Solution 6i

When $n=1$: $\text{LHS} = \sum_{r=1}^{1}(3r+1)(r+2)$

$$= (3\times1+1)(1+2)$$
$$= 12$$

$$\text{RHS} = 1(1+2)(1+3)$$
$$= 12$$

$\therefore$ LHS = RHS when $n=1$

$\therefore$ True for $n=1$

Assume true when $n=k$

$\therefore$ $n=k$ $\quad \sum_{r=1}^{k}(3r+1)(r+2) = k(k+2)(k+3)$

When $n=k+1$

$$\text{LHS} = \sum_{r=1}^{k+1}(3r+1)(r+2)$$

$$= \underbrace{(3(k+1)+1)((k+1)+2)}_{(k+1)\text{st term}} + \underbrace{\sum_{r=1}^{k}(3r+1)(r+2)}_{\text{Apply assumption for } n=k \text{ here}}$$

$$= (3k+4)\underline{(k+3)} + k(k+2)\underline{(k+3)}$$

common factor

$$= [(3k+4)+k(k+2)](k+3)$$
$$= (k^2+2k+3k+4)(k+3)$$
$$= (k^2+5k+4)(k+3)$$

factorise → $= (k+1)(k+4)(k+3)$

re-order → $= (k+1)(k+3)(k+4)$

$$= (k+1)((k+1)+2)((k+1)+3) \quad \therefore \text{true for } n=k+1$$

If the statement is true for $n=k$ then it has been shown true for $n=k+1$ and, as it is true for $n=1$, the statement is true for all n.

Solution 6ii

When $n=1$, $f(1)=4^1+5^1+6^1=15$

This is divisible by 15.

So the statement is true for $n=1$.

Assume true for $n=k$ (k odd)

So $f(k)=4^k+5^k+6^k$ is divisible by 15

$\Rightarrow f(k)=15c$ for some $c \in \mathbb{Z}$

Now consider $n=k+2$ (the next odd number)

$$\begin{aligned}
\text{So } f(k+2) &= 4^{k+2}+5^{k+2}+6^{k+2} \\
&= 16\times 4^k + 25\times 5^k + 36\times 6^k \\
&= 16\times 4^k + 16\times 5^k + 9\times 5^k + 16\times 6^k + 20\times 6^k \\
&= 16(4^k+5^k+6^k) + 9\times 5^k + 20\times 6^k \\
&= 16\times(15c) + 3\times 3\times 5\times 5^{k-1} + 4\times 5\times 3\times 2\times 6^{k-1} \\
&= 16\times 15c + 15\times 3\times 5^{k-1} + 15\times 8\times 6^{k-1} \\
&= 15(16c + 3\times 5^{k-1} + 8\times 6^{k-1})
\end{aligned}$$

Hence statement is true for $n=k+2$.

If the statement is true for $n=k$ (where k is odd) then it has been shown true for $n=k+2$ and, as it is true for $n=1$, the statement is true for all odd values of n.