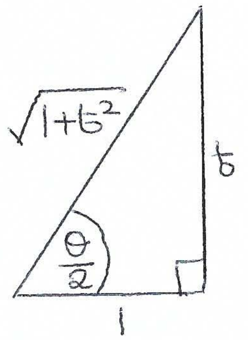
Deriving t-formulae



Let $t = \tan\left(\frac{\theta}{2}\right)$ pg 88

Use double angle formulae to prove:

$$\boxed{\sin\theta = \frac{2t}{1+t^2}} \quad \boxed{\cos\theta = \frac{1-t^2}{1+t^2}} \quad \boxed{\tan\theta = \frac{2t}{1-t^2}}$$

Example pg 88 (finding values)

Suppose $\tan\left(\frac{\theta}{2}\right) = 3$, $0 \leq \theta < 2\pi$, then using the formulae above, we can show that:

$\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$

θ is in the 2nd (top left) quadrant as only θ is positive

Example (Prove Identities) pg 89

Prove $\tan\left(\frac{\theta}{2}\right) = \frac{\tan\theta}{\sec\theta + 1}$

$$RHS = \frac{\left(\frac{2t}{1-t^2}\right)}{\left(\frac{1}{\left(\frac{1-t^2}{1+t^2}\right)}\right)+1} = \ldots = LHS$$

simplify

Example pg 89 (Prove Identities)

Use substitution $t = \tan\left(\frac{x}{2}\right)$ to prove $\frac{\boxed{\sin x} - \boxed{\cos x} + \boxed{1}}{\boxed{\sin x} + \boxed{\cos x} - \boxed{1}} = \frac{\sin x + 1}{\cos x}$

$$LHS = \frac{\boxed{\left(\frac{2t}{1+t^2}\right)} - \boxed{\left(\frac{1-t^2}{1+t^2}\right)} + \boxed{\left(\frac{1+t^2}{1+t^2}\right)}}{\boxed{\left(\frac{2t}{1+t^2}\right)} + \boxed{\left(\frac{1-t^2}{1+t^2}\right)} - \boxed{\left(\frac{1+t^2}{1+t^2}\right)}} = \ldots = RHS$$

Example (Solve equations by switching to t) pg 89

Solve $2\boxed{\cos x} + \boxed{\sin x} = 1$, $0 \leq x \leq 2\pi$

Write in terms of t:

$$2\boxed{\left(\frac{1-t^2}{1+t^2}\right)} + \boxed{\frac{2t}{1+t^2}} = 1 \Rightarrow \ldots \Rightarrow 3t^2 - 2t - 1 = 0$$

$$\Rightarrow t = 1 \text{ or } t = -\frac{1}{3} \Rightarrow \tan\frac{x}{2} = 1 \text{ or } -\frac{1}{3}$$

$$\Rightarrow x = \frac{\pi}{2}, x = 5.64$$

Check also for solutions where $\tan\left(\frac{x}{2}\right)$ has asymtotes

$2\cos\pi + \sin\pi = -2 + 0 = -2 \neq 1 \Rightarrow \pi$ isn't a solution

Example (Hidden half angles) pg 90

Show $\frac{d}{dx}\left(2 + \cos x + 15\sin\left(\frac{x}{2}\right)\right) = \frac{(1-t^2)(15t^2 - 8t + 15)}{2(1+t^2)^2}$, $t = \tan\frac{x}{4}$

Differentiate first:

$$\frac{d}{dx}\left(2 + \cos x + 15\sin\left(\frac{x}{2}\right)\right) = \boxed{-\sin x} + \frac{15}{2}\cos\frac{x}{2}$$

double angle formula

$$= \boxed{-2\boxed{\sin\left(\frac{x}{2}\right)}\boxed{\cos\left(\frac{x}{2}\right)}} - \frac{15}{2}\cos\left(\frac{x}{2}\right)$$

$$= -2\boxed{\left(\frac{2t}{1+2t^2}\right)}\boxed{\left(\frac{1-t^2}{1+t^2}\right)} - \frac{15}{2}\left(\frac{1-t^2}{1+t^2}\right) = \ldots = RHS$$

$\left(t = \tan\frac{x}{4}\right)$

Weierstrass Substitution

Using Integral Substitution with t-formulas

Use substitution

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\arctan t$$

$$\Rightarrow \frac{dx}{dt} = \frac{2}{1+t^2} \Rightarrow \boxed{dx = \frac{2}{1+t^2}\,dt}$$

formula booklet

Example pg 90 (Integral Substitution with t-formulae)

By using appropriate substitution, find $\int \frac{1}{1-\boxed{\sin x}+\boxed{\cos x}}\boxed{dx} = I$

$$I = \int\left(\frac{1}{1-\boxed{\left(\frac{2t}{1+t^2}\right)}_{\sin x}+\boxed{\left(\frac{1-t^2}{1+t^2}\right)}_{\cos x}}\right)\boxed{\left(\frac{2}{1+t^2}\right)dt}_{dx} = \dots = \int \frac{1}{1-t}\,dt = -\ln|1-t| + c$$

$$= -\ln\left|1-\tan\left(\frac{x}{2}\right)\right| + c$$

Example pg 91 (Integrals with limits)

Evaluate $\int_{\boxed{-\pi/3}}^{\boxed{\pi/3}} \boxed{\sec\theta}\,\boxed{d\theta} = \int_{\boxed{-\frac{\sqrt{3}}{3}}}^{\boxed{\frac{\sqrt{3}}{3}}} \boxed{\frac{1+t^2}{1-t^2}}_{\sec\theta}\,\boxed{\left(\frac{2}{1+t^2}\right)dt}_{d\theta}$

New limits:

$\tan\left(\frac{\boxed{\left(\frac{\pi}{3}\right)}}{2}\right) = \tan\frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{3}}$

$\tan\left(\frac{\boxed{-\frac{\pi}{3}}}{2}\right) = \tan\left(-\frac{\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{3}}$

formula booklet

$$= \dots = \int_{-\frac{\sqrt{3}}{3}}^{\frac{\sqrt{3}}{3}} \frac{2}{1-t^2} = \left[\ln\left|\frac{1+t}{1-t}\right|\right]_{-\frac{\sqrt{3}}{3}}^{\frac{\sqrt{3}}{3}} = 2\ln\left|\frac{3+\sqrt{3}}{3-\sqrt{3}}\right|$$

Taylor Series (generalise idea of Maclaurin Series)

Maclaurin series in powers of x about $x=0$

Taylor series in powers of $(x-c)$ about $x=c$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Example detailed example given on page 92

Find the Taylor series of $\sin x$ about $x=\pi$ up to and including term in $(x-\pi)^3$.

Final solution: $\sin x = -(x-\pi) + \frac{1}{6}(x-\pi)^3 + \ldots$

Example pg 93

ⓐ Find Tayor series of e^x upto and including $(x-1)^3$

ⓑ Use part a to approx. $e^{1.3}$

ⓒ What is the error ~~in~~ approx?

ⓐ $e^x = e\left(1 + (\boxed{x}-1) + \frac{(\boxed{x}-1)^2}{2} + \frac{(\boxed{x}-1)^3}{6} + \ldots\right)$

details in book

ⓑ Approximate by substituting $\boxed{x=1.3}$

$$e^{1.3} \approx e\left(1 + (\boxed{1.3}-1) + \frac{(\boxed{1.3}-1)^2}{2} + \frac{(\boxed{1.3}-1)^3}{6}\right) = 1.3495e$$

ⓒ Actual value of $e^{1.3}$ is $3.669296\ldots$

Error $= 3.669296\ldots - 1.3495e = 0.000975$ (3sf)

Example pg 93

By constructing an appropriate Taylor series for for $f(x) = \sqrt{x}$, find an approximation for $\sqrt{9.5}$

$$f(x) = 3 + \frac{(x-9)}{6} - \frac{(x-9)^2}{216} + \frac{(x-9)^3}{3888} + \ldots$$

$$\Rightarrow \sqrt{9.5} \approx 3 + \frac{0.5}{6} - \frac{0.5^2}{216} + \frac{0.5^3}{3888} = 3.08221 \text{ (5 dp)}$$

this approximation is correct to 5 d.p. it is accurate because x was sufficiently 'close' to c

Indeterminate forms

How do we deal with expressions in the form

$$\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{0}{0}$$

A possible solution: use known series expansions to find solutions

Example pg 94

$$\lim_{x\to 0} \frac{1-\cos x}{x^2 e^{2x}} = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right)}{x^2\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\dots\right)} =$$

$$= \dots = \lim_{x\to 0} \frac{\frac{1}{2!}-\frac{x^2}{4!}+\dots}{1+2x+\frac{2^2x^2}{2!}+\dots} = \frac{\frac{1}{2!}-0+\dots}{1+0+0+0+\dots} = \frac{1}{2}$$

L'Hospital's Rule

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are both 0 or $\pm\infty$, then

c is real or $\pm\infty$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

For examples, see pages 94 & 95 pgs 94 & 95

Leibnitz's Theorem

Leibitz's Theorem is an extension of Product Rule

For functions u and v, the n^{th} derivative of uv is:

$$(uv)^{(n)}(x) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}(x)$$

Note: $f^{(0)}(x) = f(x)$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Example pg 96

Calculate 2^{nd} derivative of $x^3 \ln x$ using Leibnitz's Theorem

$u(x) = x^3$ $\quad$ $v(x) = \ln(x)$

$u^{(0)}(x) = x^3$ $\quad$ $v^{(0)}(x) = \ln x$

$u^{(1)}(x) = 3x^2$ $\quad$ $v^{(1)}(x) = \frac{1}{x}$

$u^{(2)}(x) = 6x$ $\quad$ $v^{(2)}(x) = -\frac{1}{x^2}$

$$(uv)^{(2)}(x) = \binom{2}{0} u^{(0)}(x) v^{(2)}(x) + \binom{2}{1} u^{(1)}(x) v^{(1)}(x) + \binom{2}{2} u^{(2)}(x) v^{(0)}(x)$$

$$= 1 \times (x^3)\left(\frac{-1}{x^2}\right) + 2 \times (3x^2)\left(\frac{1}{x}\right) + 1 \times (6x) \ln x$$

$$= -x + 6x + 6x\ln x = x(5 + 6\ln x)$$

Examples pgs 96 and 97

Solve Differential Equations using a Series Solution

Differential equations with non-constant co-efficients $\left(\text{eg } \frac{d^2y}{dx^2} = x\frac{dy}{dx} + y\right)$ can be solved using a Taylor Series.

non-constant co-efficient

Taylor's series solution $y(x)$, for DE at $x=c$:

$$y(x) = y(c) + y'(c)(x-c) + \frac{y''(c)}{2!}(x-c)^2 + \frac{y'''(c)}{3!}(x-c)^3 + \dots$$

NB: General term is $\frac{y^{(n)}(c)}{n!}(x-c)^n$

If $c=0$ we get Maclaurin Series

Example pg 98 with an additional example on pg 99 ($c \neq 0$)

Find a series solution of DE $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$ up to term in x^4, given that $x=0, y=1$ and $\frac{dy}{dx} = 0$.

$\therefore$ We need $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$

Rewrite equation: $\boxed{y''} = \boxed{xy'} + \boxed{y}$ ①

product rule

$\Rightarrow \boxed{y'''} = \boxed{xy'' + y'} + \boxed{y'}$

$\Rightarrow y''' = xy'' + 2y'$ ②

product rule

$\Rightarrow y^{(4)} = xy''' + y'' + 2y''$

$\Rightarrow y^{(4)} = xy''' + 3y''$ ③

Initial conditions are $y(0) = 1$ and $y'(0) = 0$

Substitute in equations

① $\Rightarrow y''(0) = 0 \times y'(0) + y(0) = 1$

② $\Rightarrow y'''(0) = 0 \times y''(0) + 2y'(0) = 0$

③ $\Rightarrow y^{(4)}(0) = 0 \times y'''(0) + 3y''(0) = 3$

Co-efficients are: $y(0) = \boxed{1}$ $y'(0) = \boxed{0}$, $y''(0) = \boxed{1}$, $y'''(0) = \boxed{0}$, $y^{(4)}(0) = \boxed{3}$

Series solution is $y = \boxed{1} + \boxed{0}x + \frac{\boxed{1}}{2!}x^2 + \frac{\boxed{0}}{3!}x^3 + \frac{\boxed{3}}{4!}x^4 + \dots$

Infinitely more terms

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \dots$$

You can find the error if you know exact solution & subtract

Example pg 99

Use a Substitution to Transform a DE into something Solvable. Three steps:

1. Use given substitution to transform DE into a new one with a new variable.
2. Solve (eg using AE or IF)
3. Give general soluton of original equation by using substitution again.

Example

Use substitution $\omega = 1+x-y$ to solve $\boxed{\frac{dy}{dx} = (1+x-y)^2}$.

① $\omega = 1+x-y \Rightarrow \frac{d\omega}{dx} = 1 - \boxed{\frac{dy}{dx}} = 1 - \boxed{\omega^2}$

② Solve $\frac{d\omega}{dx} = 1-\omega^2 \Rightarrow \int \frac{d\omega}{1-\omega^2} = \int dx$

$\Rightarrow \operatorname{arctanh}(\omega) = x + C$

$\Rightarrow \boxed{\omega} = \tanh(x+c)$

③ $\Rightarrow \boxed{1+x-y} = \tanh(x+C)$

$\Rightarrow y = 1+x-\tanh(x+c)$

Example (including Auxiliary Equation Method) pg 100

Example (including Integrating Factor Method) pg 100

Example (including Auxiliary Equation Method) pg 101

Recall: Auxiliary Equation Method

$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0$ AE is $\lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow (\lambda-1)(\lambda-3) = 0$

$\Rightarrow \lambda = 1$ or $\lambda = 3$

$\Rightarrow y = Ae^{3t} + Be^{t}$

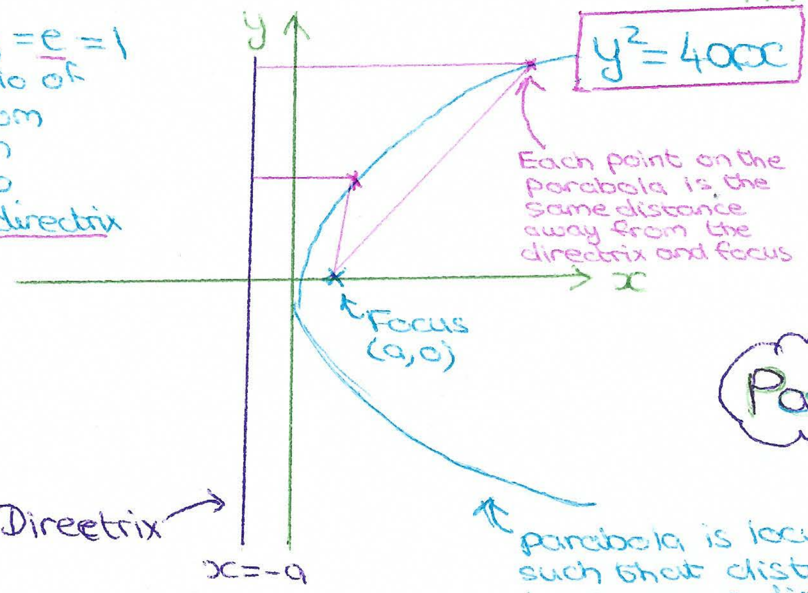
Integrating Factor Method

$v' - \frac{5}{x}v = x^2$. Multiply through by: $I(x) = e^{\int -\frac{5}{x}dx} = e^{-5\ln x} = x^{-5}$

$\Rightarrow x^{-5}v = \int x^{-5}x^2\,dx = \int x^{-3}dx = -\frac{1}{2}x^{-2} + C$

Conics can be described as parabolas, ellipses and hyperbolas. These are different types of curve you could make if you cut through a solid cone. One can describe conics with Cartesian equations, parametric equations or as a locus of points.

Eccentricity $= e = 1$
This is ratio of distance from any pt P on parabola to focus and directrix



FB ✓ $\boxed{y^2 = 4ax}$ Cartesian Eqn

Equation
$(y-h)^2 = 4a(x-k)$
shifts parabola by h up ↑ and k right →

Parabolas

Parametric Equations : $\boxed{x = at^2, \quad y = 2at}$ FB ✓

Example pg 102
Focus $(\frac{1}{2}, 0)$ and Directrix $x + \frac{1}{2} = 0$ has eqn. $y^2 = 2x$

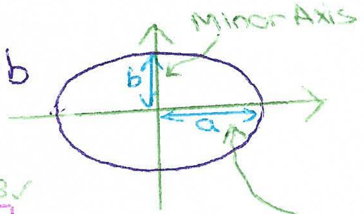
Example pg 102
Equation $\boxed{y^2 = 12(x-1)}$ has focus $(4,0)$ and directrix $x = -2$ (3+1, −3+1)
This can be deduced from:
$\boxed{y^2 = 12x}$ has focus $\boxed{(3,0)}$ and directrix $\boxed{x = -3}$
$\boxed{y^2 = 12(x-1)}$ is a translation $\boxed{1}$ to the right of $\boxed{y^2 = 12x}$

Ellipses

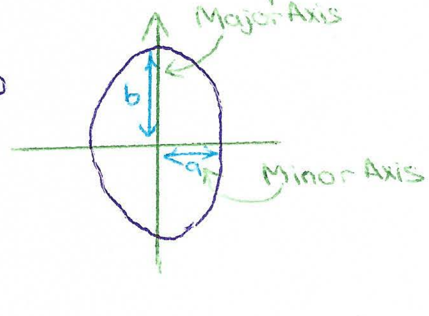
Ellipses

FB ✓ $a > b$

Cartesian Eqn $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$



FB ✓
Parametric Eqns: $\boxed{x = a\cos t, \; y = b\sin t}$

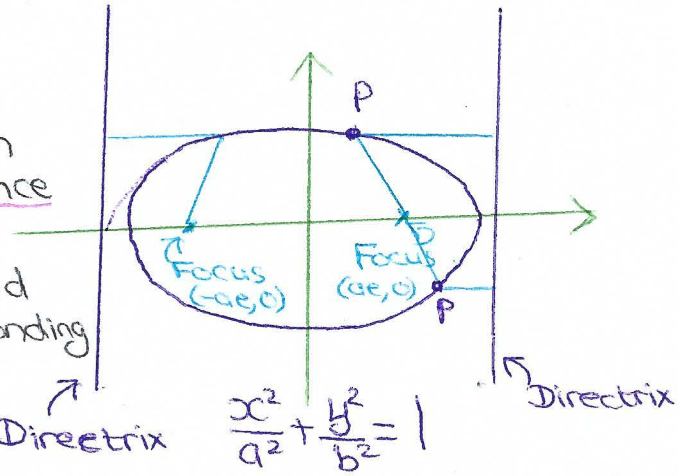


Example details of calculations pg 103 $a < b$
Ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ has parametric eqns $x = 6\cos\theta, y = 4\sin\theta$
At point $P(0, y_1)$ $(y_1 > 0)$, $y_1 = 4\sin\theta$

Ellipses have two Foci and two Directrices

Ellipse is locus of points $P(x,y)$ for which ratio of distance from a point to a focus and point corresponding to directrix is a constant e, $0<e<1$

(e is the eccentricity)



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

FB ✓

If $a>b$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has foci $(\pm ae,0)$ and directrices $x=\pm\frac{a}{e}$ with $b^2=a^2(1-e^2)$

IN FORMULA BOOKLET

If $a<b$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has foci $(0,\pm be)$ and directrices $y=\pm\frac{b}{e}$ with $a^2=b^2(1-e^2)$

Example pg 103

For ellipse $2x^2+8y^2=8$,

$$\frac{x^2}{4}+y^2=1$$ $a=2$, $b=1$ $(a>b)$

$b^2=a^2(1-e^2)$

$\Rightarrow 1=4(1-e^2)$

$\Rightarrow e=\frac{\sqrt{3}}{2}$

$(\pm ae,0)$

Foci are: $\left(\pm 2\times\frac{\sqrt{3}}{2},0\right)=(\sqrt{3},0)$ and $(-\sqrt{3},0)$

Directrices are at $x=\frac{\pm\,2/\sqrt{3}}{2}=\pm\frac{4}{\sqrt{3}}=\frac{4\sqrt{3}}{3}$ and $-\frac{4\sqrt{3}}{3}$

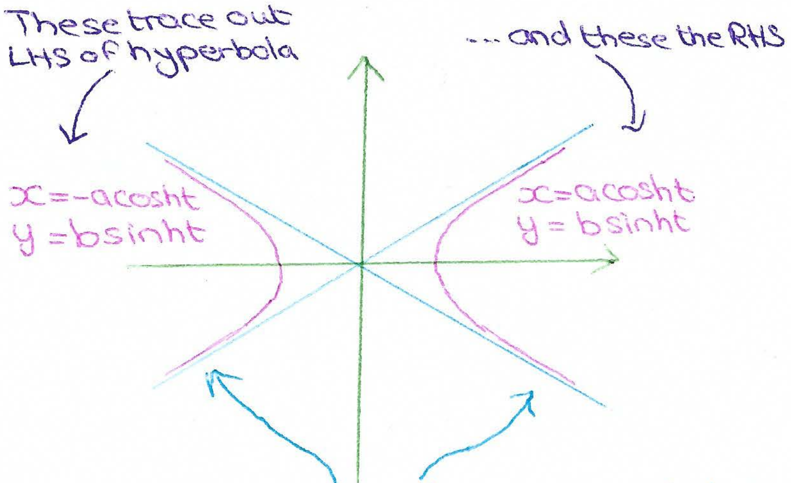
$x=\frac{-a}{e}$

Hyperbolas are related to Hyperbolic Functions

For a hyperbola,
Cartesian equation: $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ FB ✓

Parametric equations: $\boxed{x = a\sec t,\ y = b\tan t}$ FB ✓

OR → $\boxed{x = \pm a\cosh t,\ y = b\sinh t}$ FB ✓



x, y large $\Rightarrow \dfrac{x^2}{a^2} \approx \dfrac{y^2}{b^2}$ ⇒

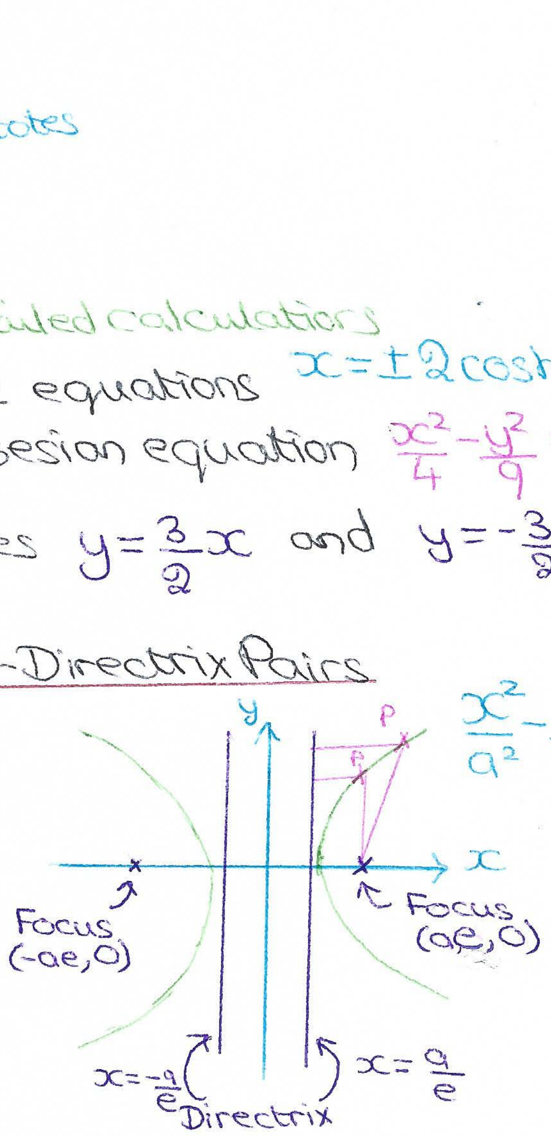
Example pg 104 gives detailed calculations

A hyperbola with parametric equations $x = \pm 2\cosh t$ and $y = 3\sinh t$ has Cartesian equation $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

Equations of asymptotes $y = \frac{3}{2}x$ and $y = -\frac{3}{2}x$

Hyperbolas have two Focus-Directrix Pairs

A hyperbola is the locus of points $P(x,y)$ for which ratio of the distance from a point to a focus and point to the corresponding directrix is a constant e, $e > 1$



$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$b^2 = a^2(e^2 - 1)$

Example pg 104

A hyperbola C has equation $\frac{x^2}{100} - \frac{y^2}{25} = 1$

$\Rightarrow \boxed{a=10}$, $\boxed{b=5}$. Since $b^2 = a^2(e^2-1)$, we have $5^2 = 10^2(e^2-1)$
$\Rightarrow \boxed{e = \frac{\sqrt{5}}{2}}$

Foci at $(ae, 0)$ and $(-ae, 0)$

$\Rightarrow$ Foci at $\left(\frac{10\sqrt{5}}{2}, 0\right)$ and $\left(-\frac{10\sqrt{5}}{2}, 0\right)$

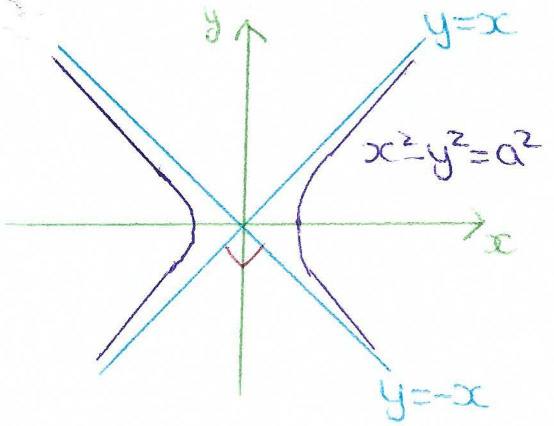
$\Rightarrow$ Foci at $(5\sqrt{5}, 0)$ and $(-5\sqrt{5}, 0)$

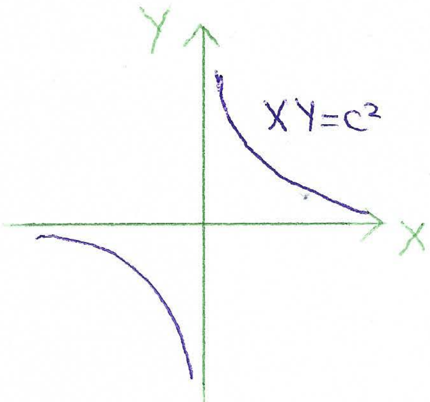
Directrices at $x = \pm\frac{a}{e}$

$\Rightarrow$ Directrices at $x = \pm\frac{a}{e} = \frac{10}{\sqrt{5}/2} = \pm 4\sqrt{5}$

Special Case: Rectangular Hyperbolas have Perpendicular Asymptotes

When $a=b$ in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $x^2 - y^2 = a^2$





Let $X = \frac{x+y}{\sqrt{2}}$

$Y = \frac{x-y}{\sqrt{2}}$

This allows you to rotate the graph so the asymptotes become the 'new' axes

Cartesian Eqn $\boxed{xy = c^2}$

Parametric Eqns $\boxed{x = ct,\ y = \frac{c}{t}}$

✓FB Pg105

Can be derived from foci and directrices of general parabola with a=b, using the above co ord transformation

Eccentricity $= e = \sqrt{2}$

Foci at $(\pm\sqrt{2}c, \pm\sqrt{2}c)$

Directrices at $x+y = \pm\sqrt{2}c$

Tangents and Normals to Curves

Use Implicit Differentiation or the Chain Rule for Conics

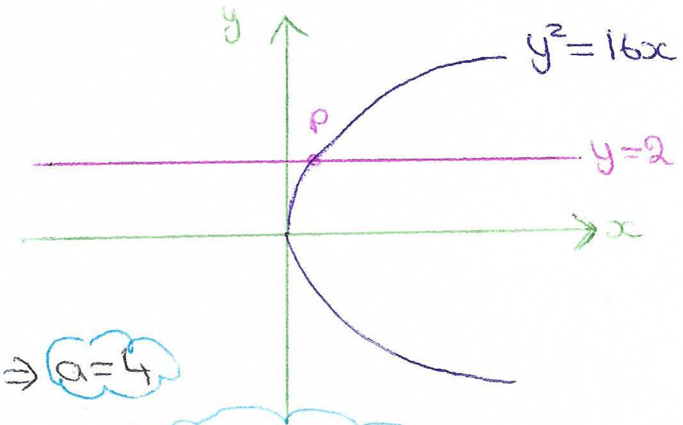
Example pg 104

a) For parabola with equation $y^2 = 4ax$, use implicit differentiation to find $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

$$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

b) If parabola C has eqn $y^2 = 16x$ and the line $y = 2$ crosses C at point P.
Find the eqn of the normal to C at P.



At point P, $y = 2$, $y^2 = 16x$

$\Rightarrow \quad 4 = 16x$

$\Rightarrow \quad x = \frac{1}{4}$

Compare eqn of C with general eqn in a): $4a = 16 \Rightarrow a = 4$

At $P(\frac{1}{4}, 2)$, gradient of C is $\frac{dy}{dx} = \frac{2 \times 4}{2} = 4$

Normal has gradient $-1 \div 4 = -\frac{1}{4}$

Equation of normal:

$$y - \boxed{y_1} = m(x - \boxed{x_1})$$

$$\Rightarrow y - \boxed{2} = -\frac{1}{4}\left(x - \boxed{\tfrac{1}{4}}\right) \Rightarrow y = -\frac{1}{4}x + \frac{33}{16}$$

Example pg106 Finding tangent to ellipse at a point

Example pg106 Finding eqn for normal to hyperbola at a point.

Example pg107 Gradient of line formed by connecting two points on an ellipse.

Deriving Cartesian Equations for Conics from Loci

Example Pg 108

Parabola = locus of pts same distance from a focus and a directrix

Show: Cartesian eqn for parabola with focus $F=(a,0)$ and directric D at $x=-a$ is $y^2=4ax$.

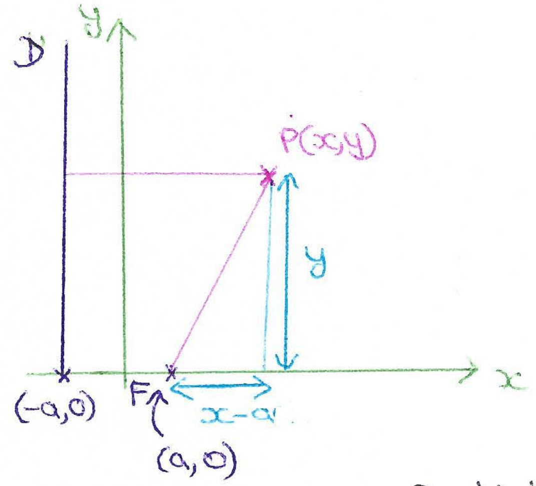
Let $P(x,y)$ be generic pt on parabola

$PF=\sqrt{(x-a)^2+y^2}$, $PD=x+a$

$|PF|=|PD|$

$\Rightarrow \sqrt{(x-a)^2+y^2}=x+a$

$\Rightarrow \dots \Rightarrow y^2=4ax$



Example Pg 108

Ellipse = locus of pts for which ratio of distance from a point to a focus and point to a directrix is a constant e, $0<e<1$.

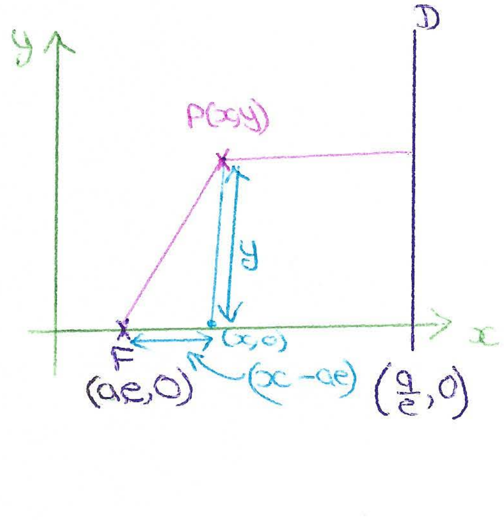
Show: Ellipse with focus $F=(ae,0)$ and directrix $x=\frac{a}{e}$ has Cartesian eqn $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$

Let $P(x,y)$ be generic pt on ellipse.

$PF=\sqrt{(x-ae)^2+y^2}$ $PD=\frac{a}{e}-x$

$\frac{PF}{PD}=e \Rightarrow \sqrt{(x-ae)^2+y^2}=\frac{a}{e}-x$

$\Rightarrow \dots \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$



Loci Problems

Loci problems Might involve chord

Example pg 108

An ellipse has eqn $x^2 + \frac{y^2}{4} = 1$.

Chord PQ of ellipse has gradient -1.

As P and Q vary, locus of the midpoint M of chord is a straight line.

Find eqn for this locus.

① Chord : straight line with gradient -1.
Let eqn of chord PQ be $\boxed{y = -x + c}$

② Endpoints of chord satisfy both eqn of chord and eqn of ellipse,
Substitute eqn $y = -x + c$ into eqn of ellipse, x-co-ordinates of endpoints are roots of

eqn:

$$x^2 + \frac{(\boxed{-x+c})^2}{4} = 1 \Rightarrow \ldots \Rightarrow 5x^2 - 2cx + c^2 - 4 = 0.$$

③ Let x_1, x_2 be roots of this quadratic,
let $M = (X, Y) \Rightarrow X = \frac{x_1 + x_2}{2}$.

For a quadratic, $ax^2 + bx + c = 0$,

sum of roots $= \frac{-b}{a}$

$$\therefore \; x_1 + x_2 = \frac{2c}{5} \Rightarrow X = \frac{c}{5} \Rightarrow c = 5X$$

④ Midpoint (X, Y) lies on chord, so $Y = -X + c$
Substituting in the expression for c gives:

$$Y = -X + 5X \Rightarrow Y = 4X$$

Example pg 109

Use Trig Identities to Eliminate Parameters

Hyperbola C with eqn $\frac{x^2}{q^2} - y^2 = 1$

Finding Vector Product

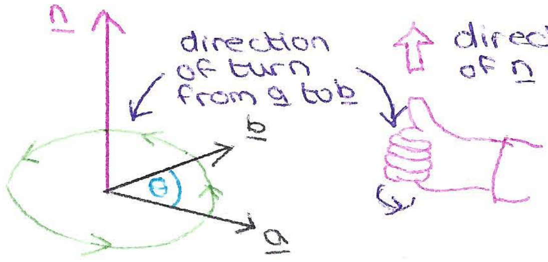
$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}|(\sin\theta)\,\hat{\underline{n}}$

θ = angle between vectors
$\hat{\underline{n}}$ = unit vector perpendicular to both $\underline{a}$ and $\underline{b}$

So $\underline{a} \times \underline{b}$ is perpendicular to $\underline{a}$ and $\underline{b}$

Direction of Vector $\hat{\underline{n}}$

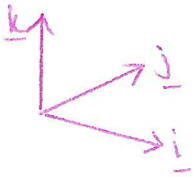
$\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$



$\underline{i} \times \underline{j} = \underline{k} \quad \underline{j} \times \underline{k} = \underline{i} \quad \underline{k} \times \underline{i} = \underline{j}$

$\underline{j} \times \underline{i} = -\underline{k} \quad \underline{k} \times \underline{j} = -\underline{i} \quad \underline{i} \times \underline{k} = -\underline{j}$

$\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$



Parallel Vectors have a Vector Product of Zero

$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{n}}$

$\theta = 180^\circ \Rightarrow \sin\theta = 0$

$\boxed{\underline{a} \times \underline{b} = 0 \Leftrightarrow \underline{a} \text{ and } \underline{b} \text{ are parallel}}$

Vector Product is Determinant of a 3x3 matrix

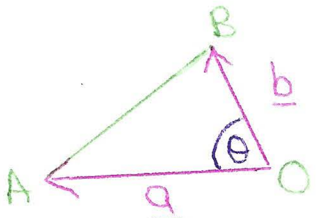
Let $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$\Rightarrow \underline{a} \times \underline{b} = \underline{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$\Rightarrow \underline{a} \times \underline{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$

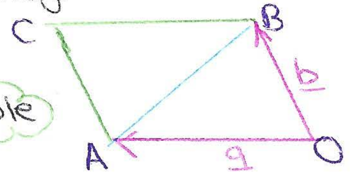
Finding Areas using Vector Product

Area of $\triangle$ $= \frac{1}{2}|\underline{a} \times \underline{b}|$



Example
pg 111

Area of Parallelogram $= |\underline{a} \times \underline{b}|$

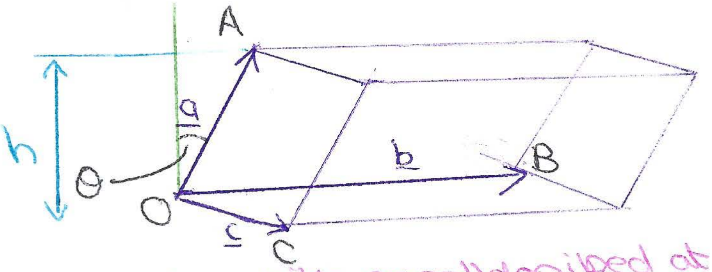


Example
pg 111

Scalar Triple Product

Volume of Parallelopiped

Volume $= |\underline{a} \cdot (\underline{b} \times \underline{c})|$



Example pg 112 (simple example with parallelopiped at O)

Example pg 112

Find volume of parallelopiped with vertex at $\boxed{(3,5,3)}$ connected to vertices at points $\boxed{(-2,-1,6)}$, $\boxed{(-1,4,3)}$ and $\boxed{(-4,-7,5)}$

vertex (this is not at O so we need to subtract it)

Let $\underline{a} = \begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} \Rightarrow \underline{a} = \begin{pmatrix} -5 \\ -6 \\ 3 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} \Rightarrow \underline{b} = \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix}$

$\underline{c} = \begin{pmatrix} -4 \\ -7 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} \Rightarrow \underline{c} = \begin{pmatrix} -7 \\ -12 \\ 2 \end{pmatrix}$

$$V = |\underline{a} \cdot (\underline{b} \times \underline{c})| = \left\| \begin{matrix} -5 & -6 & 3 \\ -4 & -1 & 0 \\ -7 & -12 & 2 \end{matrix} \right\|$$

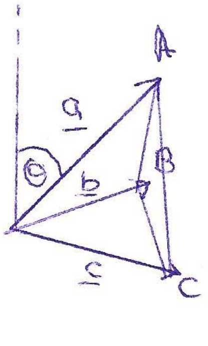
$$= \left| -5 \begin{vmatrix} -1 & 0 \\ -12 & 2 \end{vmatrix} - (-6) \begin{vmatrix} -4 & 0 \\ -7 & 2 \end{vmatrix} + 3 \begin{vmatrix} -4 & -1 \\ -7 & 12 \end{vmatrix} \right|$$

$$= \left| -5((-1)(2) - 0) + 6(-8 + 0) + 3((-4)(12) - 7) \right|$$

$$= |10 - 48 + 3(41)| = 85$$

Volume of Tetrahedron

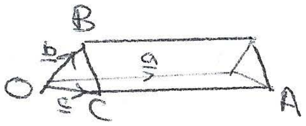
$V = \frac{1}{6} |\underline{a} \cdot (\underline{b} \times \underline{c})|$



Example pg 113 (Volume of Tetrahedron)

Triangular Prism

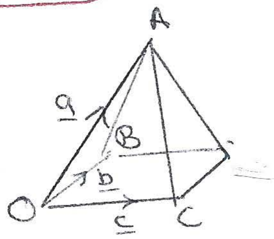
Area OBC $= \frac{1}{2}|\underline{b} \times \underline{c}|$



Volume $= \frac{1}{2}|\underline{a} \cdot (\underline{b} \times \underline{c})|$

Rectangular Based Pyramid

$V = \frac{1}{3}|\underline{a} \cdot (\underline{b} \times \underline{c})|$



Direction Ratios and Direction Cosines inform you if Two Lines are Parallel

Let $\underline{r}_1 = \underline{c} + \lambda \underline{a}$, $r_2 = \underline{d} + \mu \underline{b}$ } vector eqn of straight lines

$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\boxed{\underline{r}_1 \text{ and } \underline{r}_2 \text{ are parallel if } \underline{a} \times \underline{b} = \underline{0}.}$

But

$$\begin{pmatrix} a_2 b_3 - b_2 a_3 \\ a_1 b_3 - b_1 a_3 \\ a_1 b_2 - b_1 a_2 \end{pmatrix} = \underline{a} \times \underline{b}$$

$$\Rightarrow \frac{a_2}{a_3} = \frac{b_2}{b_3}, \quad \frac{a_1}{a_3} = \frac{b_1}{b_3}, \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

So $\underline{r}_1$ and $\underline{r}_2$ are parallel if $a_1 : a_2 : a_3 = b_1 : b_2 : b_3$. These are the direction ratios of $\underline{r}_1$ and $\underline{r}_2$

Or

$\boxed{r_1 \text{ and } r_2 \text{ are parallel if } \frac{\underline{a}}{|\underline{a}|} = \pm \frac{\underline{b}}{|\underline{b}|}}$

ie $$\frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \pm \frac{1}{\sqrt{b_1^2 + b_2^2 + b_3^2}} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Direction cosines of $\underline{r}_1$ and $\underline{r}_2$ are the components in the unit vectors, because

$$\left(\frac{a_1}{|\underline{a}|}, \frac{a_2}{|\underline{a}|}, \frac{a_3}{|\underline{a}|} \right) = (\cos\theta_x, \cos\theta_y, \cos\theta_z)$$

$\theta_x, \theta_y, \theta_z$ are the angles between $\underline{r}$ and the +ve x, y, z axes

Example pg 114

Find direction ratios of lines: $\underline{r}_1 = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1/\sqrt{3} \\ -2 \\ 0 \end{pmatrix}$ and $\underline{r}_2 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -\sqrt{3} \\ 6 \\ 0 \end{pmatrix}$ and state if the lines are parallel or not.

Direction ratio of $\underline{r}_1 = \frac{1}{\sqrt{3}} : -2 : 0 = 1 : -2\sqrt{3} : 0$

Direction ratio of $\underline{r}_2 = \frac{-1}{\sqrt{3}} : 6 : 0 = 1 : -2\sqrt{3} : 0$

$\Rightarrow$ $\underline{r}_1$ and $\underline{r}_2$ parallel

Equation for Straight Line can be written using Vector Products

Vector form of equation of straight line is

$$\underline{r} = \underline{a} + \lambda \underline{b}$$

or

$$(\underline{r} - \underline{a}) \times \underline{b} = 0$$

$\underline{a}$ is position vector, $\underline{b}$ is direction vector

Example pg 114

Points $\begin{pmatrix}1\\-3\\2\end{pmatrix}$ and $\begin{pmatrix}-4\\4\\1\end{pmatrix}$ lie on line l.

Find equation for l in the form $(\underline{r}-\underline{a})\times\underline{b} = 0$

Let $\underline{a} = \boxed{\begin{pmatrix}1\\-3\\2\end{pmatrix}}$, $\underline{b} = \begin{pmatrix}1\\-3\\2\end{pmatrix} - \begin{pmatrix}-4\\4\\1\end{pmatrix} = \boxed{\begin{pmatrix}5\\-7\\1\end{pmatrix}}$

So $(\underline{r} - \boxed{\underline{a}}) \times \boxed{\underline{b}}$

$$\Rightarrow \left(\underline{r} - \boxed{\begin{pmatrix}1\\-3\\2\end{pmatrix}}\right) \times \boxed{\begin{pmatrix}5\\-7\\1\end{pmatrix}} = 0$$

Use Vector Products to Convert between Equations of a Plane

$\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$ has $\underline{r}\cdot\underline{n} = p$, $\underline{n}$ = normal to plane

Vector Equation Scalar Equation

$$\underline{r}\cdot(\underline{b}\times\underline{c}) = p$$

$\underline{a}\cdot(\underline{b}\times\underline{c})$

Example

Find an eqn ~~of~~ in the form $\underline{r}\cdot\underline{n} = p$ for the plane

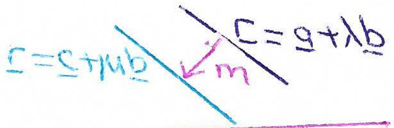
$$\underline{r} = \underset{\underline{a}}{\begin{pmatrix}2\\-1\\5\end{pmatrix}} + \lambda \underset{\underline{b}}{\begin{pmatrix}1\\1\\-3\end{pmatrix}} + \mu \underset{\underline{c}}{\begin{pmatrix}-2\\4\\1\end{pmatrix}}$$

$$\underset{\underline{b}}{\begin{pmatrix}1\\1\\-3\end{pmatrix}} \times \underset{\underline{c}}{\begin{pmatrix}-2\\4\\1\end{pmatrix}} = \ldots = \underset{\underline{b}\times\underline{c}}{\begin{pmatrix}13\\5\\6\end{pmatrix}} \Rightarrow \underline{r}\cdot \underset{\underline{b}\times\underline{c}}{\begin{pmatrix}13\\5\\6\end{pmatrix}} = \underset{\underline{a}}{\begin{pmatrix}2\\-1\\5\end{pmatrix}} \cdot \underset{\underline{b}\times\underline{c}}{\begin{pmatrix}13\\5\\6\end{pmatrix}} = 51$$

$$\Rightarrow \underline{r}\cdot\begin{pmatrix}13\\5\\6\end{pmatrix} = 51$$

Perpendicular Distance Between Two Parallel Lines

$\underline{r} = \underline{a} + \lambda \underline{b}$ and $\underline{r} = \underline{c} + \mu \underline{b}$



are parallel $\Rightarrow$ $\boxed{\text{Shortest distance} = \left|\frac{(\underline{c}-\underline{a})\cdot\underline{b}}{|\underline{b}|}\right|}$

$|\underline{m}|$

Example pg 115

Line l_1 contains points $\boxed{(1,5,5)}$ and $\boxed{(2,-4,0)}$

Line l_2 is parallel to l_1 and contains point $\boxed{(-1,-3,1)}$. Find perpendicular distance between l_1 and l_2.

Use $\underline{b} = \begin{pmatrix}2\\-4\\0\end{pmatrix} - \begin{pmatrix}1\\5\\5\end{pmatrix} = \begin{pmatrix}1\\-9\\-5\end{pmatrix} \Rightarrow |\underline{b}| = \sqrt{107}$,

$$\underline{c}-\underline{a} = \begin{pmatrix}-1\\-3\\1\end{pmatrix} - \begin{pmatrix}1\\5\\5\end{pmatrix} = \begin{pmatrix}-2\\-8\\-4\end{pmatrix}$$

$$\Rightarrow (\underline{c}-\underline{a})\times\underline{b} = \begin{vmatrix} i & j & k \\ -2 & 8 & -4 \\ 1 & -9 & -5 \end{vmatrix} = \begin{pmatrix}4\\-14\\26\end{pmatrix}$$

$$\Rightarrow |(\underline{c}-\underline{a})\times\underline{b}| = \sqrt{4^2+14^2+26^2} = 2\sqrt{222}$$

$$\Rightarrow \frac{|(\underline{c}-\underline{a})\times\underline{b}|}{|\underline{b}|} = \frac{2\sqrt{222}}{\sqrt{107}}$$

Perpendicular distance between Two Skew Lines

$\boxed{\underline{r} = \underline{a} + \lambda\underline{b},\ \underline{r} = \underline{c} + \mu\underline{d} \text{ is } \left|\frac{(\underline{c}-\underline{a})\cdot(\underline{b}\times\underline{d})}{|\underline{b}\times\underline{d}|}\right|}$

Example Pg 115

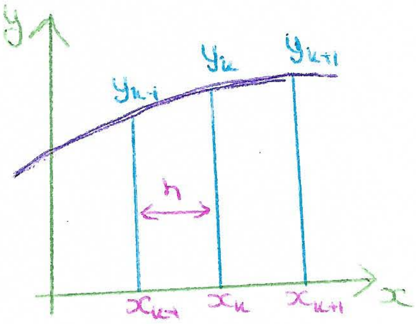
Find perpendicular distance between $\underline{r} = \underbrace{\begin{pmatrix}-7\\-4\\9\end{pmatrix}}_{a} + \lambda \underbrace{\begin{pmatrix}-8\\6\\9\end{pmatrix}}_{b}$

and $\underline{r} = \underbrace{\begin{pmatrix}8\\2\\13\end{pmatrix}}_{c} + \mu \underbrace{\begin{pmatrix}-2\\3\\5\end{pmatrix}}_{d}$

$\underline{b}\times\underline{d} = \begin{pmatrix}-8\\6\\9\end{pmatrix}\times\begin{pmatrix}-2\\3\\5\end{pmatrix} = \begin{pmatrix}3\\22\\-12\end{pmatrix} \Rightarrow |\underline{b}\times\underline{d}| = \sqrt{3^2+22^2+12^2} = 7\sqrt{13}$

$\underline{c}-\underline{a} = \begin{pmatrix}8\\2\\13\end{pmatrix} - \begin{pmatrix}-7\\-4\\9\end{pmatrix} = \begin{pmatrix}15\\6\\4\end{pmatrix} \Rightarrow (\underline{c}-\underline{a})\cdot(\underline{b}\times\underline{d}) = \begin{pmatrix}15\\6\\4\end{pmatrix}\cdot\begin{pmatrix}3\\22\\-12\end{pmatrix} = 129 \Rightarrow \left|\frac{(\underline{c}-\underline{a})\cdot(\underline{b}\times\underline{d})}{|\underline{b}\times\underline{d}|}\right| = \frac{129}{7\sqrt{13}}$

Numerical Methods are needed for DEs which you can't solve exactly



Method 1

Finding y_{k+1} using y_k

$$\boxed{y_{k+1} \approx y_k + h y'(x_k)}$$

Example pg 116 (details of calculation in book)
Use step length $h = 0.1$ to approximate $y(1.2)$ for $y'(x) = \sin(x+y)$, given $x=1, y=1$

Method 2

Find y_{k+1} using y_k and y_{k-1}

$$\boxed{y_{k+1} \approx y_{k-1} + 2h y'(x_k)}$$

Example pg 116 (details of calculations in book)
Use step length $h = 0.01$ to approximate $y(1.02)$ for $\frac{dy}{dx} = \frac{y(x+1)}{x+y}$ given $x=1, y=1$

Numerical Methods for Second Order DEs

$$\boxed{y_{k+1} = 2y_k - y_{k-1} + h^2 y''(x_k)}$$

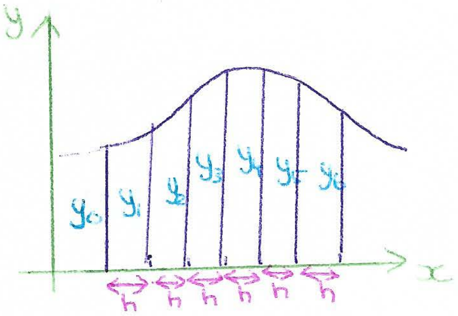
Example pg 117 (details of calculations in book
Use step length $h=0.2$ to approx. $y(1.4)$ for $\frac{d^2y}{dx^2} = x\cos y$ given at $x=1, y=0$ and at $x=1.2, y=0.1$

Example pg 117

Simpson's Rule

Simpson's Rule to approx. area under a curve

$$\int_a^b y = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + y_n\right)$$



$$\int_a^b y = \frac{h}{3}\left(\text{first} + \text{last} + 4y_{2k+1} + 2y_{2k}\right)$$

where n is an even number of strips or intervals and $h = \frac{b-a}{n}$

Example pg 118 (Plus extra example pg 119 ; to improve approx, increase strips)

Use Simpson's Rule to find an approx value for $\int_1^2 \frac{1}{x+2}\,dx$ using 4 strips. Give answer to 3 d.p.

Width: $h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$

x-values: $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$

Table

x	$y = \frac{1}{x+2}$
$x_0 = 1$	$y_0 = \frac{1}{3}$
$x_1 = 1.25$	$y_1 = \frac{1}{3.25}$
$x_2 = 1.5$	$y_2 = \frac{1}{3.5}$
$x_3 = 1.75$	$y_3 = \frac{1}{3.75}$
$x_4 = 2$	$y_4 = \frac{1}{4}$

Formula

$$\int_1^2 \frac{1}{x+2}\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right]$$

$$= \frac{0.25}{3}\left[\frac{1}{3} + 4\left(\frac{1}{3.25} + \frac{1}{3.75}\right) + \frac{2}{3.5} + \frac{1}{4}\right]$$

$$= 0.288 \text{ (3dp)}$$

Calculate Error (if required)

Exact answer $= \left[\ln(x+2)\right]_1^2 = \ln 4 - \ln 3 = 0.288$ (3dp)

Error is 0 to 3 dp. $\Rightarrow$ Simpson's rule gives good approx.

Algebraic Inequalities

For algebraic inequalities, add and subtract to get zero on one side

Steps to solve inequalities:
1. all terms on one side, 0 on the other
2. find critical values where $f(x)=0$
3. either sketch graph or test values on either side of c.v. to see where inequality is true.

Example pg 120

$x^2(x+3) > 4(x+3)$

$\Rightarrow \quad x^2(x+3) - 4(x+3) > 0$

$\Rightarrow \ldots \Rightarrow (x+3)(x+2)(x-2) > 0$

CV: $x=-3$, $x=-2$, $x=2$

Graph +ve for $-3 < x < -2$ or $x > 2$

Multiply by Square of all Denominators

Example pg 120

$$\frac{x}{x-4} \geqslant \frac{2}{x+1} \quad \Rightarrow \quad \frac{x}{x-4}(x-4)^2(x+1)^2 \geqslant \frac{2}{x+1}(x-4)^2(x+1)^2$$

Read text book for complete example

Squaring helps with modulus signs

If $|f(x)| > |g(x)| \Rightarrow |f(x)|^2 > |g(x)|^2$

Example pg 121

$|x+2| \geqslant |x^2-4| \Rightarrow (x+2)^2 \geqslant (x^2-4)^2$

Use Graphs to find solutions of $|f(x)| > g(x)$

Example pg 121

$|x+2| \geqslant x^2-4$