1. (a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -k < x < k$$

stating the value of the constant k.

(5)

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2-3x}\right)$$

(5)

Solution 1a

definition

Let $y = \tanh^{-1}x \Rightarrow x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

$\sinh y = \frac{1}{2}(e^y - e^{-y})$

$\cosh y = \frac{1}{2}(e^y + e^{-y})$

$\Rightarrow x = \frac{e^{2y} - 1}{e^{2y} + 1}$

multiplying top and bottom by e^y

$\Rightarrow x(e^{2y} + 1) = e^{2y} - 1$

multiplying out $\Rightarrow xe^{2y} + x = e^{2y} - 1$

rearranging $\Rightarrow xe^{2y} - e^{2y} = -1 - x$

$\Rightarrow e^{2y}(x-1) = -1-x$

$\Rightarrow e^{2y} = \frac{-1-x}{x-1} = \frac{1+x}{1-x}$

taking ln (to find y) $\Rightarrow \ln e^{2y} = \ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow 2y \overset{1}{\ln e} = \ln\left(\frac{1+x}{1-x}\right)$

dividing by 2 $\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

To ~~get~~ find k, note $\frac{1+x}{1-x} > 0$.

$\Rightarrow 1+x > 0$ (if $1-x > 0$) $\Rightarrow 1 > x$

$\Rightarrow x > -1$

$\Rightarrow -1 < x < 1$

Solution 1b

Since part b mentions that you can solve it by using part a, try to spot similarities in the two equations. In particular $2x$. Replacing x by $2x$ in part a gives

$$\tanh^{-1}(2x) = \frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right) \quad (*)$$

$$\text{Part b} \Rightarrow 2x = \tanh(\ln\sqrt{2-3x}) \quad (**)$$

$$(*) \Rightarrow 2x = \tanh\left(\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)\right)$$

$$= \tanh\left(\ln\left(\frac{1+2x}{1-2x}\right)^{1/2}\right)$$

$(**)$ & last line

$$\Rightarrow 2x = \tanh\boxed{(\ln\sqrt{2-3x})} = \tanh\boxed{\left(\ln\left(\frac{1+2x}{1-2x}\right)^{1/2}\right)}$$

$$\Rightarrow \boxed{\ln\sqrt{2-3x}} = \boxed{\ln\left(\frac{1+2x}{1-2x}\right)^{1/2}}$$

$$\Rightarrow \sqrt{2-3x} = \left(\frac{1+2x}{1-2x}\right)^{1/2}$$

squaring

$$\Rightarrow 2-3x = \frac{1+2x}{1-2x}$$

multiplying out

$$\Rightarrow 1+2x = (2-3x)(1-2x)$$

expanding

$$\Rightarrow 1+2x = 2-4x-3x+6x^2$$

tidying

$$\Rightarrow 6x^2-9x+1=0 \xRightarrow{\text{solving quadratic}} x = \frac{9 \pm \sqrt{(-9)^2-4\times6\times1}}{2\times6}$$

solving

$$\Rightarrow x = \frac{9-\sqrt{57}}{12}$$

2. The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are p, q and r.

Without solving the equation, find the value of

(i) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

(8)

Solution 2i

Recall: If root of equation $ax^3+bx^2+cx+d=0$ are α, β and γ, then
$\alpha+\beta+\gamma = -b/a$
$\alpha\beta+\alpha\gamma+\beta\gamma = c/a$ and
$\alpha\beta\gamma = -d/a$

$$\frac{2}{p}+\frac{2}{q}+\frac{2}{r}$$

$$= \frac{2\boxed{(qr+pr+pq)}}{\boxed{pqr}} \quad (*)$$

In $x^3-2x^2+4x-5=0$,

$$\Rightarrow \boxed{qr+pr+pq = \frac{4}{1}}, \quad \boxed{pqr = \frac{5}{1}}$$

$$(*) \Rightarrow \frac{2}{p}+\frac{2}{q}+\frac{2}{r} = \frac{2\boxed{(4)}}{\boxed{5}} = \frac{8}{5}$$

Solution 2ii

expanding

$$(p-4)(q-4)(r-4) = (pq-4p-4q+16)(r-4)$$

expanding

$$= pqr - 4pq - 4pr + 16p - 4qr + 16q + 16r - 16\times 4$$

tidying

$$= pqr - 4\boxed{(pq+pr+qr)} + 16\boxed{(p+q+r)} - 64$$

$p+q+r = \frac{-(-2)}{1}$ from equation

$$= 5 - 4\boxed{(4)} + 16\boxed{(2)} - 64$$

$$= -43$$

Solution 2iii

$$p^3+q^3+r^3 = \boxed{(p+q+r)}^3 - 3\boxed{(p+q+r)}\boxed{(pq+pr+qr)} + 3\boxed{pqr}$$

$$= \boxed{2}^3 - 3 \times \boxed{(2)}\boxed{(4)} + 3\boxed{(5)}$$

$$= 8 - 24 + 15$$

$$= -1$$

3.

$$f(x) = \frac{1}{\sqrt{4x^2 + 9}}$$

(a) Using a substitution, that should be stated clearly, show that

$$\int f(x)\mathrm{d}x = A\sinh^{-1}(Bx) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence find, in exact form in terms of natural logarithms, the mean value of f(x) over the interval [0, 3].

(2)

Solution 3a

$\int f(x)\,dx$

NB: $\sqrt{r^2+x^2} \Rightarrow x = r\sinh u$

$\sqrt{x^2-r^2} \Rightarrow x = r\cosh u$

or use clue in answer

$$I = \int \frac{1}{\sqrt{4x^2+9}}\,dx = \int \frac{1}{\sqrt{9\left(\frac{4x^2}{9}+1\right)}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\frac{4}{9}x^2+1}}\,dx$$

in the form $\sqrt{r^2+1}$ Then let $y = \operatorname{arcsinh} r$

Let $\boxed{y = \operatorname{arcsinh}\frac{2x}{3}} \Rightarrow \frac{2x}{3} = \sinh y$

$\Rightarrow x = \frac{3}{2}\sinh y$

$\Rightarrow \frac{dx}{dy} = \frac{3}{2}\cosh y$

$\Rightarrow$ "$dx = \boxed{\frac{3}{2}\cosh y\,dy}$"

So $I = \frac{1}{3}\int \frac{1}{\sqrt{\frac{4}{9}x^2+1}}\boxed{dx} = \frac{1}{3}\int \frac{1}{\sqrt{\sinh^2 y+1}} \times \boxed{\frac{3}{2}\cosh y\,dy}$

$= \frac{1}{3}\int \frac{1}{\cosh y} \times \frac{3}{2}\cosh y\,dy = \frac{1}{2}\int dy = \frac{1}{2}\boxed{y} + c$

$= \frac{1}{2}\boxed{\operatorname{arcsinh}\frac{2x}{3}} + c$

Solution 3b

$$\frac{1}{3-0}\int_0^3 f(x)\,dx = \frac{1}{3}\left[\frac{1}{2}\operatorname{arcsinh}\frac{2x}{3}\right]_0^3$$

$$= \frac{1}{6}\left(\operatorname{arcsinh}\frac{2(3)}{3} - \operatorname{arcsinh}\frac{2(0)}{3}\right)$$

$$= \frac{1}{6}\left(\operatorname{arcsinh} 2 - 0\right)$$

$\sinh^{-1} z = \ln\left(z+\sqrt{1+z^2}\right)$

$$= \frac{1}{6}\left(\ln\left(2+\sqrt{1+2^2}\right)\right)$$

$$= \frac{1}{6}\ln\left(2+\sqrt{5}\right)$$

4. The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \dots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \dots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

Solution 4a

$$C+iS = (\cos\theta + i\sin\theta) + \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 9\theta + i\sin 9\theta) + \frac{1}{8}(\cos 13\theta + i\sin 13\theta) + \dots$$

$$= (\cos\theta + i\sin\theta) + \frac{1}{2}(\cos\theta + i\sin\theta)^5 + \frac{1}{2^2}(\cos\theta + i\sin\theta)^9 + \frac{1}{2^3}(\cos\theta + i\sin\theta)^{13} + \dots$$ (by De Moivre)

$$= r + \frac{1}{2}r^5 + \frac{1}{2^2}r^9 + \frac{1}{2^3}r^{13} + \dots$$ where $r = \cos\theta + i\sin\theta$

$$= r\left(1 + \frac{1}{2}r^4 + \frac{1}{2^2}r^8 + \frac{1}{2^3}r^{12} + \dots\right)$$

Let $p = \frac{1}{2}r^4$

Sum of geometric series $\sum_{k=0}^{\infty} p^k$ which converges is: $\frac{1}{1-p}$

$$= r(1 + p + p^2 + p^3 + \dots)$$

$$= r \cdot \frac{1}{1-p} = r \cdot \frac{1}{1 - \frac{1}{2}r^4} = \frac{2r}{2 - r^4}$$

$$= \frac{2(\cos\theta + i\sin\theta)}{2 - (\cos\theta + i\sin\theta)^4} = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

Solution 4b

Split eqn into real and imaginary parts.
Then equate imaginary parts on both sides to find S.

$$C + i\boxed{S} = \frac{2(\cos\theta + i\sin\theta)}{2-\cos 4\theta - i\sin 4\theta} \times \frac{(2-\cos 4\theta + i\sin 4\theta)}{(2-\cos 4\theta + i\sin 4\theta)}$$

multiply by complex conjugate to get a real no. in the denominator

$$= \frac{2(c_\theta + i s_\theta)(2 - c_{4\theta} + i s_{4\theta})}{(2 - c_{4\theta} - i s_{4\theta})(2 - c_{4\theta} + i s_{4\theta})}$$

$$= \frac{\boxed{2}(2c_\theta - c_\theta c_{4\theta} + \boxed{i c_\theta s_{4\theta} + 2 i s_\theta - i s_\theta c_{4\theta}} - s_\theta s_{4\theta})}{\boxed{(2-c_{4\theta})(2-c_{4\theta}) + s_{4\theta}^2}}$$

this is now real

Equating imaginary parts:

$$S = \boxed{\frac{2(c_\theta s_{4\theta} + 2s_\theta - s_\theta c_{4\theta})}{4 - 4c_{4\theta} + c_{4\theta}^2 + s_{4\theta}^2}}$$

$\cos^2\theta + \sin^2\theta = 1$

$$= \frac{2(2\sin\theta + \boxed{\sin 4\theta\cos\theta - \cos 4\theta\sin\theta})}{4 - 4\cos 4\theta + 1}$$

$$= \frac{2(2\sin\theta + \boxed{\sin(4\theta - \theta)})}{5 - 4\cos 4\theta}$$

$$= \frac{4\sin\theta + \boxed{2\sin 3\theta}}{5 - 4\cos 4\theta}$$

5. An engineer is investigating the motion of a sprung diving board at a swimming pool.
Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.

(2)

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of 55 cm s^{-1}.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E.

(8)

(c) Comment on the suitability of the model for large values of t.

(2)

Solution 5a

Auxiliary Equation $4m^2+4m+37=0$

$$\Rightarrow m = \frac{-4 \pm \sqrt{4^2-4(4)(37)}}{8} = \frac{-4\pm\sqrt{576}\,i}{8} = \frac{-4\pm 24i}{8}$$

$$\Rightarrow m = -\frac{1}{2} + 3i$$

$$\Rightarrow h = e^{-0.5t}(A\cos 3t + B\sin 3t) \quad (*)$$

Solution 5b

From part a,

$$(**) \quad \frac{dh}{dt} = -0.5e^{-0.5t}(A\cos 3t + B\sin 3t) + e^{-0.5t}(-3A\sin 3t + 3B\cos 3t)$$

From (*) and given $t=0, h=-20$, we have

$$-20 = A$$

From (**) and given $t=0$, $h=-20$, $\frac{dh}{dt}=55$, we have:

$$55 = -0.5(-20+0)+(3B) \Rightarrow B = 15$$

So $h = e^{-0.5t}(-20\cos 3t + 15\sin 3t)$

Solution 5b continued.

Max vertical displacement occurs when $\frac{dh}{dt} = 0$. So substituting this and $A=-20$, $B=15$ in $(**)$, we have

$$0 = -0.5e^{-0.5t}(-20\cos 3t + 15\sin 3t) + e^{-0.5t}(60\sin 3t + 45\cos 3t)$$

$$\Rightarrow 0 = e^{-0.5t}(10\cos 3t - 7.5\sin 3t) + e^{-0.5t}(60\sin 3t + 45\cos 3t)$$

$\times$ through by $e^{0.5t}$:

$$\Rightarrow 10\cos 3t - 7.5\sin 3t + 60\sin 3t + 45\cos 3t = 0$$

$$\Rightarrow 55\cos 3t + 52.5\sin 3t = 0$$

$$\Rightarrow \cancel{-\sin 3t =} \quad 55\cos 3t = -52.5\sin 3t$$

$$\Rightarrow \tan 3t = \frac{-55}{52.5}$$

$$\Rightarrow 3t = \tan^{-1}\left(\frac{-55}{52.5}\right) \Rightarrow t = 0.778\,s$$

Solution 5c

Candidate can say:

- Value of h is very small when t is large and this is likely to be correct; $\Rightarrow$ model is appropriate

OR

- When t is large, value of h is never equal to 0 $\Rightarrow$ model is not appropriate for large values of t.

6. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2i$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + iy$, where x and y are real and exact.

(6)

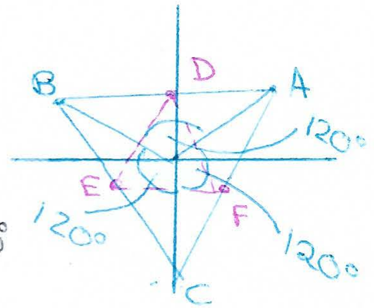
The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

Solution 6a

Try to visualise first



Each point is a rotation of the previous one of 120° about the origin

$$\begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6\left(-\frac{1}{2}\right) + 2\left(-\frac{\sqrt{3}}{2}\right) \\ 6\left(\frac{\sqrt{3}}{2}\right) + 2\left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} -3-\sqrt{3} \\ 3\sqrt{3}-1 \end{pmatrix}$$

$\Rightarrow B = (-3-\sqrt{3}) + (3\sqrt{3}-1)i$

$$\begin{pmatrix} \cos 240° & -\sin 240° \\ \sin 240° & \cos 240° \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{6}{2} + \sqrt{3} \\ -6\frac{\sqrt{3}}{2} - 1 \end{pmatrix} = \begin{pmatrix} -3+\sqrt{3} \\ -3\sqrt{3}-1 \end{pmatrix}$$

$\Rightarrow C = (-3+\sqrt{3}) + (-3\sqrt{3}-1)i$

In the diagram, note that Area($\triangle DEF$) = ($\frac{1}{4}$ $\triangle ABC$) Area

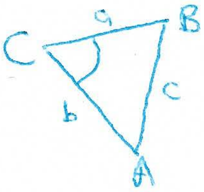
Also Area $\triangle AOB = \frac{1}{3}$ Area $\triangle ABC$

Use these facts to ~~prove~~ solve part b

Solution 6b

In diagram in part a,

Recall : Area of triangle $= \frac{1}{2}ab\sin C$



Area of $\triangle AOB$

$= \frac{1}{2}|a||b|\sin(\text{Angle at Origin})$

$= \frac{1}{2}\sqrt{2^2+6^2}\sqrt{2^2+6^2}\sin 120^\circ$

$= \frac{1}{2}\sqrt{40}\sqrt{40}\,\frac{\sqrt{3}}{2} = \sqrt{3}\times 10.$

We also know that:

$\text{Area}(\triangle DEF) = \frac{\text{Area}(\triangle ABC)}{4} = \frac{3}{4}\text{Area}(\triangle AOB)$

$= \frac{3}{4}\times\sqrt{3}\times 10$

$= \frac{15}{2}\sqrt{3}$

7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix **M** has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$2x - y + z = p$$
$$3x - 6y + 4z = 1$$
$$3x + 2y - z = 0$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$2x - y + z = 1$$
$$3x - 5y + 4z = q$$
$$3x + 2y - z = 0$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

Solution 7a

M has an inverse $\Leftrightarrow \det M \neq 0$

$$\det M = 2\begin{vmatrix} k & 4 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & k \\ 3 & 2 \end{vmatrix}$$

$$= 2(-k-8) + (-3-12) + (6-3k)$$

$$= -2k - 16 - 15 + 6 - 3k$$

$$= -5k - 25$$

$\det M \neq 0 \Leftrightarrow -5k - 25 \neq 0 \Rightarrow k \neq -5$

Solution 7b

$$2x - y + z = p$$
$$3x - 6y + 4z = 1 \qquad k=-6$$
$$3x + 2y - z = 0$$

$$\Rightarrow \begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \qquad (*)$$

This is the matrix in part a with $k = -6$

$$\text{Adj}\, M = \begin{pmatrix} \begin{vmatrix} -6 & 4 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 3 & -6 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ -6 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & -6 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 6-8 & -(-3-12) & (6+18) \\ -(1-2) & (-2-3) & -(4+3) \\ (-4+6) & -(8-3) & (-12+3) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 15 & 24 \\ 1 & -5 & -7 \\ 2 & -5 & -9 \end{pmatrix}$$

$$(\text{Adj}\, M)^T = \begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix}$$

From part a, $\det M = -5k - 25$
in part b, $k = -6$

$$\Rightarrow M^{-1} = \frac{1}{\det M}\begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix} = \frac{1}{30-25}\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$\det M = -5(-6) - 25 = 30 - 25$

$$= \frac{1}{5}\begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix}$$

$$(*) \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & +1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix}\begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \Rightarrow \frac{1}{5}\begin{pmatrix} -2p+1 \\ 15p-5 \\ 24p-7 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Solution 7ci

From part a, recall that M does 'not' have an inverse if $k=-5$. In this part, $k=-5$.

Allocate a number to one variable and solve for other two.

Let $x=0$ and substitute in equations:

$$\left.\begin{array}{l} 2(0)-y+z=1 \\ 3(0)-5y+4z=q \\ 3(0)+2y-z=0 \end{array}\right\} \Rightarrow \begin{array}{ll} -y+z=1 & (1) \\ -5y+4z=q & (2) \\ 2y-z=0 & (3) \end{array}$$

Sub (1) in (3):

$$2(z-1)-z=0 \Rightarrow 2z-2-z=0$$
$$\Rightarrow z=2$$
$$(1) \Rightarrow -y+2=1$$
$$\Rightarrow y=1$$
$$(2) \Rightarrow -5(1)+4(2)=q$$
$$\Rightarrow -5+8=q$$
$$\Rightarrow q=3$$

Solution 7c ii

For this value of q, this set of simultaneous equations represents planes that intersect in a line.

8.

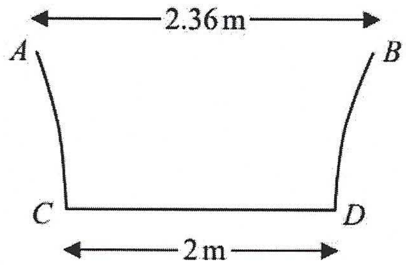


Figure 1

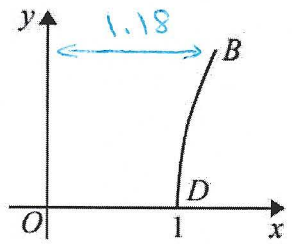


Figure 2

Figure 1 shows the central vertical cross section *ABCD* of a paddling pool that has a circular horizontal cross section. Measurements of the diameters of the top and bottom of the paddling pool have been taken in order to estimate the volume of water that the paddling pool can contain.

Using these measurements, the curve *BD* is modelled by the equation

$$y = \ln(3.6x - k) \qquad 1 \leqslant x \leqslant 1.18$$

as shown in Figure 2.

(a) Find the value of k.

(1)

(b) Find the depth of the paddling pool according to this model.

(2)

The pool is being filled with water from a tap.

(c) Find, in terms of h, the volume of water in the pool when the pool is filled to a depth of h m.

(5)

Given that the pool is being filled at a constant rate of 15 litres every minute,

(d) find, in cm h^{-1}, the rate at which the water level is rising in the pool when the depth of the water is 0.2 m.

(3)

Solution 8a

When $\boxed{x = 1}$, $y = 0$. So $\ln(3.6x - k) = \ln(3.6\boxed{(1)} - k) = 0$

$\Rightarrow \ln(3.6 - k) = 0$

$\Rightarrow 3.6 - k = 1 \Rightarrow k = 2.6$

Solution 8b

Find y when $\boxed{x = 1.18}$:

$y = \ln(3.6 \times \boxed{1.18} - 2.6) = 0.4995\ldots\text{m}$

Solution 8c

$$y = \ln(3.6x - 2.6)$$

$$\Rightarrow e^{y} = 3.6x - 2.6$$

$$\Rightarrow 3.6x = e^{y} + 2.6$$

$$\Rightarrow \quad x = \frac{e^{y} + 2.6}{3.6}$$

Now

$$V = \pi \int_0^h \left(\frac{e^{y} + 2.6}{3.6}\right)^2 dy$$

$$= \frac{\pi}{3.6^2} \int_0^h e^{2y} + 5.2e^{y} + 6.76\, dy$$

$$= \frac{\pi}{3.6^2} \left[\frac{1}{2}e^{2y} + 5.2e^{y} + 6.76y\right]_0^h$$

$$= \frac{\pi}{3.6^2}\left(\left(\frac{1}{2}e^{2h} + 5.2e^{h} + 6.76h\right) - \left(\frac{1}{2}e^{0} + 5.2e^{0} + 6.76(0)\right)\right)$$

$$= \frac{\pi}{3.6^2}\left(\frac{1}{2}e^{2h} + 5.2e^{h} + 6.76h - 5.7\right)$$

Solution 8d

$$\frac{dV}{dh} = \frac{\pi}{3.6^2}\left(e^{2h} + 5.2e^{h} + 6.76\right)$$

By chain rule,

Invert $\frac{dV}{dh}$

15 divided by 1000

1 litre = $0.001\,m^3$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{3.6^2}{\pi\left(e^{2h} + 5.2e^{h} + 6.76\right)} \times 0.015 \times 60$$

$h = 0.2\,m$

Convert mins to hours

expression in $m\,h^{-1}$

$$\Rightarrow \frac{dh}{dt} = 0.254\,m\,h^{-1}$$

$\times 100$

$$\Rightarrow \frac{dh}{dt} = 25.4\,cm\,h^{-1}$$

$\times 100$ & convert to $cm\,h^{-1}$

60 mins $= 1$ hr $\Rightarrow 60\,h^{-1} = 1\,min^{-1}$