① The transformation P is an enlargement, centre the origin, with scale factor k, where $k > 0$

The transformation Q is a rotation through angle θ degrees anticlockwise about the origin.

The transformation P followed by the transformation Q is represented by the matrix $M = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$

(a) Determine

(i) the value of k

(ii) the smallest value of θ (4)

A square S has vertices at the points with co-ordinates $(0,0)$, $(a,-a)$, $(2a,0)$ and (a,a) where a is a constant.

The square S is transformed to the square S′ by the transformation represented by M.

(b) Determine, in terms of a, the area of S′ (2)

Solution 1ai

Transformation P followed by Q is

$$\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{ROTATION}} \underbrace{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}}_{\text{ENLARGEMENT}} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

$$\text{LHS} = \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

Try taking determinant of both sides as it looks like you may get an identity like $\cos^2 + \sin^2 = 1$

Taking determinants:

$$k^2\cos^2\theta + k^2\sin^2\theta = (-4)(-4) - (-4\sqrt{3})(4\sqrt{3})$$

$$\Rightarrow k^2 = 16 + 48 = 64 \Rightarrow k = 8 \ (k > 0)$$

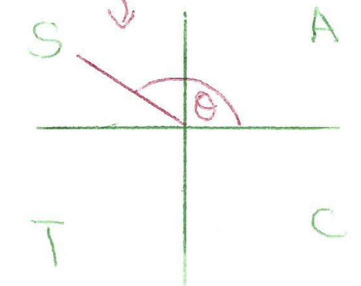
Solution 1aii

From part ai,

$$\begin{pmatrix} 8\cos\theta & -8\sin\theta \\ 8\sin\theta & 8\cos\theta \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

Hence $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$

In the CAST diagram:



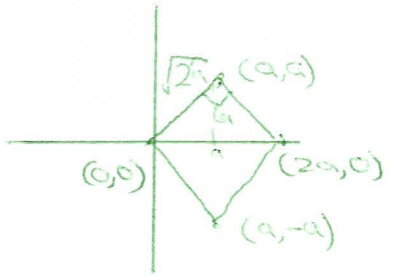
From calculator:

$\theta = 120°$

Solution 1b

Try visualising what is happening first

Sketch S



Side length of S $= \sqrt{2}a$ (using Pythagoras)

$\therefore$ Area of S $= 2a^2$

$(\sqrt{2}a)^2$

Now S' = enlargement of S scale factor k followed by rotation

So, S' has side length $k(\sqrt{2}a)$

$\therefore$ Area of $S' = [k(\sqrt{2}a)]^2 = [8(\sqrt{2}a)]^2$

$= 64 \times 2 \times a^2$

$= 128a^2$

Side note: $S'_{\text{Area}} = \det(M) S_{\text{Area}}$ is possibly a standard result

So if S' is transformation of S by M, then

Area of S' = (Area of S) det(M)

2 (a) Use the Maclaurin series expansion for $\cos x$ to determine the expansion of $\cos^2\left(\frac{x}{3}\right)$ in ascending powers of x, up to and including the term in x^4

Give each term in simplest form. (2)

(b) Use the answer to part (a) and calculus to find an approximation to 5 d.p. for

$$\int_{\pi/6}^{\pi/2} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) dx \qquad (3)$$

(c) Use the integration function on your calculator to evaluate

$$\int_{\pi/6}^{\pi/2} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) dx \qquad (1)$$

Give your answer to 5 d.p.

(d) Assuming that the calculator answer in part (c) is accurate to 5 d.p., comment on the accuracy of the approximation found in part (b) (1)

Solution 2a

$$\cos^2\left(\frac{x}{3}\right) = \left(1 - \frac{\left(\frac{x}{3}\right)^2}{2!} + \frac{\left(\frac{x}{3}\right)^4}{4!}\right)^2$$

$$= \left(1 - \frac{x^2}{18} + \frac{x^4}{1944}\right)^2$$

$$\approx 1 - \frac{x^2}{18} + \frac{x^4}{1944} - \frac{x^2}{18} + \frac{x^4}{324} + \frac{x^4}{1944}$$

Powers higher than 4 not include

$$= 1 - \frac{x^2}{9} + \frac{x^4}{243}$$

Solution 2b

$$\int_{\pi/6}^{\pi/2} \frac{1}{x}\cos^2\left(\frac{x}{3}\right)dx \approx \int_{\pi/6}^{\pi/2} \frac{1}{x}\left(1-\frac{x^2}{9}+\frac{x^4}{243}\right)dx$$

$$= \int_{\pi/6}^{\pi/2} \frac{1}{x} - \frac{x}{9} + \frac{x^3}{243}\, dx$$

$$= \left[\ln x - \frac{1}{18}x^2 + \frac{1}{972}x^4\right]_{\pi/6}^{\pi/2}$$

$$= \left(\left(\ln \pi/2 - \frac{1}{18}(\pi/2)^2 + \frac{1}{972}(\pi/2)^4\right)\right.$$
$$\left. - \left(\ln \pi/6 - 1/18(\pi/6)^2 + \frac{1}{972}(\pi/6)^4\right)\right)$$

$$= 0.98295 \quad (5\text{d.p})$$

Solution 2c

0.98280 (Calculator)

Solution 2d

$0.98280 \approx 0.983$ (3d.p)

$0.98295 \approx 0.983$ (3dp)

Answer is quite accurate and correct to 3 d.p.

③ The cubic equation

$$ax^3 + bx^2 - 19x - b = 0 \quad (*)$$

where a and b are constants, has roots α, β and γ.

The cubic equation $w^3 - 9w^2 - 97w + c = 0$ $(**)$

where c is a constant, has roots $(4\alpha - 1), (4\beta - 1)$ and $(4\gamma - 1)$.

Without solving either cubic, determine the values of a, b and c.

Solution 3

$$0 = ax^3 + bx^2 + cx + d$$
$$\sum \alpha = -\frac{b}{a}$$
$$\sum \alpha\beta = \frac{c}{a}$$
$$\alpha\beta\gamma = -\frac{d}{a}$$

$(*) \Rightarrow \boxed{\alpha + \beta + \gamma = -\frac{b}{a}}$ and

$(**) \Rightarrow (4\alpha - 1) + (4\beta - 1) + (4\gamma - 1) = 9$

$\Rightarrow 4(\alpha + \beta + \gamma) - 3 = 9$

$\Rightarrow 4\left(-\frac{b}{a}\right) = 12 \Rightarrow -\frac{b}{a} = 3 \Rightarrow \boxed{b = -3a}$

Also, $(**) \Rightarrow (4\alpha - 1)(4\beta - 1) + (4\alpha - 1)(4\gamma - 1) + (4\beta - 1)(4\gamma - 1) = -97$

$\Rightarrow (16\alpha\beta - 4\alpha - 4\beta + 1) + (16\beta\gamma - 4\beta - 4\gamma + 1) + (16\gamma\alpha - 4\gamma - 4\alpha + 1) = -97$

$\Rightarrow 16(\alpha\beta + \beta\gamma + \gamma\alpha) - 8(\alpha + \beta + \gamma) + 3 = -97$

$\Rightarrow 16\boxed{(\alpha\beta + \beta\gamma + \gamma\alpha)} - 8\boxed{(\alpha + \beta + \gamma)} = -100$

Also, $(*) \Rightarrow \boxed{\alpha\beta + \beta\gamma + \gamma\alpha = \frac{19}{a}}$

So $16 \times \boxed{\left(\frac{19}{a}\right)} - 8\boxed{\left(\frac{-b}{a}\right)} = -100$

$\Rightarrow -304 + 8b = -100a$

$\Rightarrow 76 - 2\boxed{b} = 25a$

$\Rightarrow 76 - 2\boxed{(-3a)} = 25a$

$\Rightarrow 76 + 6a = 25a$

$\Rightarrow 19a = 76$

$\Rightarrow a = 4 \Rightarrow b = -12$

Solution 3 continued

We also have (*) $\Rightarrow \boxed{\alpha\beta\gamma = \frac{b}{a}}$

$\Rightarrow \alpha\beta\gamma = -3$ (since $b=-12, a=4$)

(**) $\Rightarrow (4\alpha-1)(4\beta-1)(4\gamma-1) = -c$

EXPAND

$\Rightarrow (16\alpha\beta - 4\alpha - 4\beta + 1)(4\gamma - 1) = -c$

$\Rightarrow 64\boxed{\alpha\beta\gamma} - 16\alpha\gamma - 16\beta\gamma + 4\gamma - 16\alpha\beta + 4\alpha + 4\beta - 1 = -c$

$\Rightarrow 64\boxed{(-3)} - 16\boxed{(\alpha\beta + \beta\gamma + \gamma\alpha)} + 4\boxed{(\alpha+\beta+\gamma)} - 1 = -c$ FACTORISING

$\Rightarrow -192 - 16\boxed{\left(\frac{-19}{a}\right)} + 4\boxed{\left(\frac{-b}{a}\right)} - 1 = -c$

SUBSTITUTE
$a=4$
$b=-12$

$\Rightarrow -192 - 16\boxed{\left(\frac{-19}{4}\right)} + 4\boxed{(3)} - 1 = -c$

$\Rightarrow -105 = -c$ SOLVE for c

$\Rightarrow c = 105$

We already know $a=4$ and $b=-12$

4 (i) A is a 2 by 2 matrix and B is a 2 by 3 matrix.

Giving a reason for your answer, explain whether it is possible to evaluate

(a) AB

(b) A+B (2)

(ii) Given that

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \lambda I$$

where a, b and λ are constants

(a) determine
- the value of λ
- the value of a
- the value of b

(b) Hence deduce the inverse of the matrix $\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}$ (3)

(iii) Given that

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \quad 0 \leq \theta < \pi$$

determine the values of θ for which the matrix M is singular. (4)

Solution 4ia

It is possible to evaluate AB as the no. of columns of matrix A matches the no. of rows of matrix B. $\underset{\text{matrix}}{a \times x} \times \underset{\text{matrix}}{x \times b} = \underset{\text{matrix}}{a \times b}$

Solution 4ib

It is not possible to evaluate A+B as A and B have different dimensions, namely different no. of columns

Matrix is given as rows by columns

Solution 4iia

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$LHS = \begin{pmatrix} 0+6-1 & \cdots & \cdots \\ \cdots & 5a+0+0 & \cdots \\ 0b+2a-b & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \boxed{5} & \cdots & \cdots \\ \cdots & \boxed{5a} & \cdots \\ \boxed{2a-b} & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \boxed{\lambda} & 0 & 0 \\ 0 & \boxed{\lambda} & 0 \\ \boxed{0} & 0 & \lambda \end{pmatrix}$$

$\Rightarrow \lambda = 5$

$\Rightarrow 5a = \lambda \Rightarrow a = 1$

$\Rightarrow 2a - b = 0 \Rightarrow b = 2$

Solution 4iib

From 4iia, $\begin{pmatrix} -5 & 3 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore$ Inverse Matrix $\frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix}$

Solution 4iii

Singular Matrix has determinant 0.

$$\det M = \det\begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} = 1(\sin\theta \sin 2\theta - \cos\theta\cos 2\theta) - 1(0) + 1(0) = 0$$

$$= \cos(3\theta) \quad \text{(compound angle formula)}$$

$$= 0$$

$$\Rightarrow 3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

FM1
Paper 1: CPM1

Ref: 9FM0/01
Nov 2021

5 i Evaluate the improper integral

$$\int_1^{\infty} 2e^{-\frac{1}{2}x}\,dx \qquad (3)$$

ii The air temperature θ°C, on a particular day in London is modelled by the equation

$$\theta = 8 - 5\sin\left(\frac{\pi}{12}t\right) - \cos\left(\frac{\pi}{6}t\right) \qquad 0 \leqslant t \leqslant 24$$

where t is the number of hours after midnight

a Use calculus to show that the mean air temperature on this day is 8°C, according to the model (3)

Given that the actual mean air temperature recorded on this day was higher than 8°C,

b explain how the model could be refined. (1)

Solution 5i

Let $I = \int_1^{\infty} 2e^{-\frac{1}{2}x}\,dx$.

Let $u = -\frac{1}{2}x \Rightarrow \frac{du}{dx} = -\frac{1}{2} \Rightarrow$ "$dx = -2du$"

* Hence $\int 2e^{-\frac{1}{2}x}\,dx = \int 2e^{u}(-2)\,du = -4\int e^{u}\,du$

$= -4e^{u} + C$

So $2\int_1^{\infty} e^{-\frac{1}{2}x}\,dx = \left[-4e^{-\frac{1}{2}x}\right]_1^{\infty} = \lim_{t\to\infty}\left(-4e^{-\frac{1}{2}t}\right) - \left(-4e^{-\frac{1}{2}}\right)$

$= 0 + 4e^{-\frac{1}{2}}$

$= 4e^{-\frac{1}{2}}$

*NOTE: This is a detailed breakdown. There is a simple rule, however, when integrating a known function with a constant attached to the variable. Simply DIVIDE by the constant and integrate as usual

Solution 5iia

$$\text{Mean temperature} = \frac{1}{24}\int_0^{24}\left(\boxed{8}\boxed{-5\sin\left(\frac{\pi}{12}t\right)}\boxed{-\cos\left(\frac{\pi}{6}t\right)}\right)dt$$

NOTE! To integrate $\sin x$ is straight forward. $(-\cos x)$ To integrate $\sin px$ we simply divide by p to get: $-\frac{1}{p}\cos px$

This represents the "TOTAL" or "SUM". So when we divide this by 24 we get the "MEAN".

$$\therefore \text{Mean Temperature} = \frac{1}{24}\left[\boxed{8t}\boxed{+5\left(\frac{12}{\pi}\right)\cos\left(\frac{\pi t}{12}\right)}\boxed{-\frac{6}{\pi}\sin\left(\frac{\pi t}{6}\right)}\right]_0^{24}$$

dividing by $\pi/12$ dividing by $\pi/6$

$$= \frac{1}{24}\left[\left(8\times24+5\left(\frac{12}{\pi}\right)\cos\left(\frac{\pi}{12}\times24\right)-\frac{6}{\pi}\sin\left(\frac{\pi}{6}\times24\right)\right)-\left(0+5\left(\frac{12}{\pi}\right)\cos(0)-0\right)\right]$$

$$= \frac{1}{24}\left[\left(192+\frac{60}{\pi}\cos(2\pi)-\frac{6}{\pi}\sin(4\pi)\right)-\frac{60}{\pi}\right]$$

$$= \frac{1}{24}\left[192+\frac{60}{\pi}-0-\frac{60}{\pi}\right]$$

$$= 8$$

Solution 5iib

Given that actual mean temperature was higher, the constant in the model can be increased if we wish to refine the model.

6) A tourist decides to do a bungee jump from a bridge over a river. One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.

The tourist jumps off the bridge.

At time t seconds after the tourist reaches their lowest point, their vertical displacement is x metres, above a fixed point 30 metres vertically above the river.

When $t = 0$,

- $x = -20$
- velocity of tourist is $0\,\mathrm{m\,s^{-1}}$
- acceleration of tourist is $13.6\,\mathrm{m\,s^{-2}}$

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the equation

$$5k\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 17x = 0, \quad t \geq 0$$

where k is a positive constant.

(a) Determine the value of k. (2)

(b) Determine the particular solution to the differential equation (7)

(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point. (2)

(d) Give a limitation of the model. (1)

Solution 6a

In the equation x = distance, $\frac{dx}{dt}$ = velocity and $\frac{d^2x}{dt^2}$ = acceleration

When $t=0$, $x=-20$, $\frac{dx}{dt}=0$ and $\frac{d^2x}{dt^2}=13.6$

Substitute these values into the equation to find k:

$$5k\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+17x=0$$

Substituting gives

$$5k(13.6)+2k(0)+17(-20)=0$$

$$\Rightarrow \quad 68k=340$$

$$\Rightarrow \quad k=5$$

Solution 6b

From part a, equation is: $25\frac{d^2x}{dt^2}+10\frac{dx}{dt}+17x=0$

$\therefore$ auxiliary equation is:

$$25\lambda^2+10\lambda+17=0$$

Solving quadratic gives: $\lambda=\boxed{-0.2}\pm\boxed{0.8}i$

So $x=e^{\boxed{-0.2}t}(A\cos \boxed{0.8}t+B\sin \boxed{0.8}t)$

Recall: Differentiating $\cos x$ is simple: $(-\sin x)$
Differentiating $\cos px$ (multiply outside by constant) gives: $-p\sin px$

Differentiating by product rule $\therefore$ $\frac{dx}{dt}=e^{-0.2t}(-0.8A\sin 0.8t+0.8B\cos 0.8t)-0.2e^{-0.2t}(A\cos 0.8t+B\sin 0.8t)$

At $t=0$, $\frac{dx}{dt}=0$. So: $0=e^0(-0.8A\sin 0+0.8B\cos 0)-0.2e^0(A\cos 0+B\sin 0)$

$$\Rightarrow 0=0.8B-0.2A$$

$$\Rightarrow 2A=8B \Rightarrow \boxed{A=4B}$$

Also, at $t=0$, $x=-20$. So: $-20=e^0(A\cos(0)+B\sin(0))$

$$\Rightarrow \boxed{-20=A}$$

Hence $\boxed{B=\frac{A}{4}}=\frac{-20}{4}=-5$

$$\therefore \quad x=e^{-0.2t}(-20\cos 0.8t-5\sin 0.8t)$$

FMI
Paper 1: CP M1

Ref: 9FM0/01
Nov 2021

Solution 6c

At $t = 15$:

$$x = e^{-0.2\times15}(-20\sin(0.8\times15) - 5\sin(0.8\times15))$$

$$= -0.70668\ldots$$

Remember to take into account the 30m in the question.

Vertical height $= 30 - 0.7$

$= 29.3\text{m}$

Solution 6d

Tourist is modelled as a particle.

(other examples:

- unlikely rope will remain taut
- prediction of model is the tourist will continue to move up and down (but, in fact, they will lose momentum))

⑦ The plane Π has equation

$$\underline{r} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

where λ and μ are scalar parameters

ⓐ Show that the vector $2i + 3j - 4k$ is perpendicular to Π (2)

ⓑ Hence find a Cartesian equation of Π (2)

The line l has equation $\underline{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$

where t is a scalar parameter.

The point A lies on l.

Given that the shortest distance between A and Π is $2\sqrt{29}$.

ⓒ Determie the possible co-ordinates of A. (4)

Solution 7a

Now $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = (-2) + 6 + (-4) = 0$

and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 4 + 0 + (-4) = 0$

As $2i + 3j - 4k = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ is perpendicular to both direction vectors (two non parallel vectors) of Π then it must also be perpendicular to Π.

Solution 7b

vector perpendicular to plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

Equation of plane

position vector

$$\Rightarrow 2x + 3y - 4z = 6 + 9 - 8$$

$$\Rightarrow 2x + 3y - 4z = 7$$

Solution 7c

using formula

$$\frac{|2(4+t) + 3(-5+6t) - 4(2-3t) - 7|}{\sqrt{2^2 + 3^2 + (-4)^2}} = 2\sqrt{29}$$

$$\Rightarrow 2\sqrt{29} = \frac{|8 + 2t - 15 + 18t - 8 + 12t - 7|}{\sqrt{29}}$$

$$\Rightarrow 58 = |32t - 22|$$

$$\Rightarrow 29 = |16t - 11|$$

$$\Rightarrow 29^2 = (16t - 11)^2$$

$$\Rightarrow 0 = (16t - 40)(16t + 18)$$

$$\Rightarrow t = \frac{5}{2}, \quad t = -\frac{9}{8}$$

At $t = \frac{5}{2}$, $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 13/2 \\ 10 \\ -11/2 \end{pmatrix}$

At $t = -9/8$, $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} - \frac{9}{8}\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 23/8 \\ -47/4 \\ 43/8 \end{pmatrix}$

(8) Two different colours of paint are being mixed together in a container.
The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.
Initially, the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.
The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second.
- adding blue paint to the container at a rate of 1 litre per second.
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let r litres be the amount of red paint in the container at time t seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{dr}{dt} = 2 - \frac{r}{a}$$

where a is a constant to be determined

(8b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model.
Give your answer to the nearest second.

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer.

Solution 8a

As 3 litres of paint are added each second, and 3 litres are removed each second, the volume of paint remains constant. There is a constant 30l of paint

$$\frac{dr}{dt} = \underset{\text{2l red paint added each second}}{2} - \overset{\text{3l of paint removed each second}}{3} \times \underbrace{\frac{r}{30}}_{\text{proportion of paint that is red}}$$

So $\dfrac{dr}{dt} = 2 - \dfrac{r}{10}$

Solution 8b

$$\frac{dr}{dt} = 2 - \frac{r}{10} \Rightarrow \frac{dr}{dt} = \frac{20-r}{10}$$

$$\Rightarrow \int \frac{1}{20-r}\,dr = \int \frac{1}{10}\,dt$$

$$\Rightarrow -\ln|20-r| = \frac{1}{10}t + C$$

At $t=0$, $r=10$ (as stated in question)

So $-\ln|10| = C$ ie $C = -\ln 10$

$\therefore$ equation is $-\ln|20-r| = \frac{1}{10}t - \ln 10$

At $r=15$:

$$-\ln|20-15| = \frac{1}{10}t - \ln 10$$

$$\Rightarrow -\ln 5 = \frac{1}{10}t - \ln 10$$

$$\Rightarrow \frac{1}{10}t = \ln 10 - \ln 5$$

$$\Rightarrow \frac{1}{10}t = \ln\frac{10}{5}$$

$$\Rightarrow t = 10\ln 2$$

$\Rightarrow$ $t = 7$ seconds (nearest second)

Solution 8c

7 is close to 9 (over 20% away) so the model is not a very good model

9 (a) Use a hyperbolic substitution and calculus to show that

$$I = \int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arccosh} x\right] + k$$

where k is an arbitrary constant. (6)

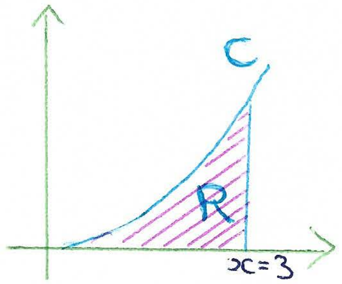


Figure 1

Curve C has equation $y = \frac{4}{15}x \operatorname{arccosh} x$ $(x \geqslant 1)$

Figure 1 shows a sketch of part of the curve C, ~~wi~~ the x-axis and the line with equation $x=3$.

(b) Using algebraic integration and the result from part (a), show that the region R is given by

$$\frac{1}{15}\left[17\ln(3+2\sqrt{2}) - 6\sqrt{2}\right]$$ (5)

Solution 9a

Since integral includes $\sqrt{x^2-1}$

use $x = \cosh u$

NB: $\sqrt{r^2+x^2} \Rightarrow x = r\sinh u$

$\sqrt{x^2-r^2} \Rightarrow x = r\cosh u$

or use clue in answer

$$\Rightarrow \frac{dx}{du} = \sinh u \Rightarrow \boxed{dx = \sinh u\,du}$$

So $$I = \int \frac{x^2}{\sqrt{x^2-1}}\boxed{dx} = \int \frac{\cosh^2 u}{\underbrace{\sqrt{\cosh^2 u - 1}}_{\sinh^2 u}} \cdot \boxed{\sinh u\,du}$$

$$= \int \frac{\cosh^2 u}{\sinh u}\sinh u\,du = \int \underbrace{\cosh^2 u}_{\frac{1}{2}(\cosh 2u + 1)}\,du$$

$\Rightarrow x = \cosh u$

$$= \frac{1}{2}\int \cosh 2u + 1\,du$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2u + u\right) + k$$

Solution 9a continued

$$I = \frac{1}{2}\left(\frac{1}{2}\sinh 2u + u\right) + k \qquad \text{where } x = \cosh u$$

But $\sinh 2u = 2\sinh u \cosh u$.

$$\therefore \quad I = \frac{1}{2}\left(\boxed{\sinh u}\boxed{\cosh u} + u\right) + k$$

Since $\boxed{x = \cosh u}$ and $\boxed{\sinh^2 u = \cosh^2 u - 1}$, we have:

$$\sinh^2 u = x^2 - 1$$

So

$$I = \frac{1}{2}\left(\boxed{\sqrt{x^2-1}} \cdot \boxed{x^2} + \operatorname{arccosh} x\right) + k$$

Solution 9b

$$\int \frac{4}{15} x \operatorname{arccosh} x \, dx$$

Let $u = \operatorname{arccosh} x$ $\quad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{\sqrt{x^2-1}}$ $\quad v = \frac{1}{2}x^2$

$$= \frac{4}{15}\left(\left[\frac{1}{2}x^2 \operatorname{arccosh} x\right]_1^3 - \int_1^3 \frac{1}{2} \times \frac{x^2}{\sqrt{x^2-1}} \, dx\right)$$

At $y = 0$,
$0 = \frac{4}{15} x \operatorname{arccosh} x$
$\Rightarrow x = 0$ OR $\operatorname{arccosh} x = 0$
So $x = 1$

from graph
not a limit

$$= \frac{4}{15}\left(\frac{1}{2}\left(\boxed{3^2 \operatorname{arcosh} 3} - \boxed{1^2 \operatorname{arcosh} 1}\right) - \frac{1}{2}\boxed{\left[\frac{1}{2}\left(x\sqrt{x^2-1} + \operatorname{arccosh} x\right)\right]_1^3}\right)$$

$$= \frac{4}{15}\left(\frac{1}{2}\left(\boxed{9 \operatorname{arccosh} 3} - \boxed{0}\right) - \frac{1}{2}\left(\boxed{\frac{1}{2}\left(\left(3\sqrt{3^2-1} + \operatorname{arccosh} 3\right) - \left(1\sqrt{1^2-1} + \operatorname{arccosh} 1\right)\right)}\right)\right)$$

$$= \frac{4}{15}\left(\frac{1}{2} \times 9 \operatorname{arccosh} 3 - \frac{1}{4}\left(\left(6\sqrt{2} + \operatorname{arccosh} 3\right) - (0 + 0)\right)\right)$$

$$= \frac{1}{15}\left(18 \operatorname{arccosh} 3 - \left(6\sqrt{2} + \operatorname{arccosh} 3\right)\right)$$

$$= \frac{1}{15}\left(17 \operatorname{arccosh} 3 - 6\sqrt{2}\right)$$

(Now convert arccosh into ln form

$$= \frac{1}{15}\left(17 \ln\left(3 + \sqrt{3^2-1}\right) - 6\sqrt{2}\right)$$

$$= \frac{1}{15}\left(17 \ln\left(3 + 2\sqrt{2}\right) - 6\sqrt{2}\right)$$