1. An ellipse has equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and eccentricity e_1

A hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity e_2

Given that $e_1 \times e_2 = 1$

(a) show that $a^2 = 3b^2$

(4)

Given also that the coordinates of the foci of the ellipse are the same as the coordinates of the foci of the hyperbola,

(b) determine the equation of the hyperbola.

(3)

Solution 1a

	Ellipse	Hyperbola
SF	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
PF	$(a\cos\theta, b\sin\theta)$	
E	$e<1$	$e>1$
	$b^2=a^2(1-e^2)$	$b^2=a^2(e^2-1)$
Foci	$(\pm ae, 0)$	$(\pm ae, 0)$
Dir	$x=\pm\frac{a}{e}$	$x=\pm\frac{a}{e}$

For ellipse:

$4 = 16(1-e_1^2)$ ①

For hyperbola

$b^2 = a^2(e_2^2-1)$ ②

① $\Rightarrow$ $3 = +4e_1^2$ ③

② $\Rightarrow$ $b^2+a^2 = a^2e_2^2$ ④

③ x ④ $\Rightarrow 3(b^2+a^2) = 4a^2e_1^2e_2^2$

$= 4a^2(e_1e_2)^2$ since $e_1e_2=1$

$= 4a^2(1)^2$ (stated in question)

$\Rightarrow 3b^2+3a^2 = 4a^2$

$\Rightarrow 3b^2 = a^2$

Solution 1b

From ③, we have $e_1 = \frac{\sqrt{3}}{2} \Rightarrow$ Focus for ellipse $= (\pm 2\sqrt{3}, 0)$

$\Rightarrow$ Focus for hyperbola $= (\boxed{\pm 2\sqrt{3}}, 0)$ since both are given as equal in the ques

But, generally focus for hyperbola is $(\boxed{\pm ae_2}, 0)$

We also ~~kno~~ know $e_1e_2 = 1$ and $e_1 = \frac{\sqrt{3}}{2} \Rightarrow e_2 = \frac{2}{\sqrt{3}}$

$\Rightarrow \boxed{\pm ae_2 = 2\sqrt{3}} \Rightarrow \pm a\left(\frac{2}{\sqrt{3}}\right) = 2\sqrt{3} \Rightarrow \boxed{a = \pm 3}$

But $3b^2 = a^2 \Rightarrow 3b^2 = 3^2 \Rightarrow \boxed{b = \pm\sqrt{3}} \Rightarrow \frac{x^2}{\boxed{9}} - \frac{y^2}{\boxed{3}} = 1$

2. During 2029, the number of hours of daylight per day in London, H, is modelled by the equation

$$H = 0.3\sin\left(\frac{x}{60}\right) - 4\cos\left(\frac{x}{60}\right) + 11.5 \qquad 0 \leqslant x < 365$$

where x is the number of days after 1st January 2029 and the angle is in radians.

(a) Show that, according to the model, the number of hours of daylight in London on the 31st January 2029 will be 8.13 to 3 significant figures.

(1)

(b) Use the substitution $t = \tan\left(\frac{x}{120}\right)$ to show that H can be written as

$$H = \frac{at^2 + bt + c}{1 + t^2}$$

where a, b and c are constants to be determined.

(2)

(c) Hence determine, according to the model, the date of the first day of 2029 when there will be at least 12 hours of daylight in London.

(4)

Solution 2a

At $x = 30$:

$$H = 0.3\sin\left(\frac{30}{60}\right) - 4\cos\left(\frac{30}{60}\right) + 11.5 = 8.13 \ (3\text{sf})$$

Solution 2b

$$\sin\left(\frac{x}{60}\right) = \frac{2t}{1+t^2} \qquad \cos\left(\frac{x}{60}\right) = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow H = 0.3 \times \frac{2t}{1+t^2} - 4 \times \left(\frac{1-t^2}{1+t^2}\right) + 11.5$$

$$= \frac{0.6t}{1+t^2} - \frac{4-4t^2}{1+t^2} + \frac{11.5+11.5t^2}{1+t^2}$$

$$= \frac{15.5t^2 + 0.6t + 7.5}{1+t^2} \Rightarrow a = 15.5,\ b = 0.6,\ c = 7.5$$

Solution 2c

$$12 \leqslant H \Rightarrow 12 \leqslant \frac{15.5t^2 + 0.6t + 7.5}{1+t^2}$$

$$\Rightarrow 12(1+t^2) \leqslant 15.5t^2 + 0.6t + 7.5$$

$$\Rightarrow 0 \leqslant 3.5t^2 + 0.6t - 4.5$$

$$\Rightarrow t = \frac{-3 \pm 12\sqrt{11}}{35} \Rightarrow \tan\left(\frac{x}{120}\right) = \frac{-3 \pm 12\sqrt{11}}{35}$$

$$\Rightarrow \frac{x}{120} = \tan^{-1}\left(\frac{-3 \pm 12\sqrt{11}}{35}\right) = 0.810\ldots,\ -0.885\ldots,\ 2.256\ldots$$

$$x = 97,\ \cancel{-106},\ \cancel{270}$$

means 1st day discard later values

out of range

$\Rightarrow$

PTO

3. With respect to a fixed origin O, the points A and B have coordinates $(2, 2, -1)$ and $(4, 2p, 1)$ respectively, where p is a constant.

For each of the following, determine the possible values of p for which,

(a) OB makes an angle of 45° with the positive x-axis

(3)

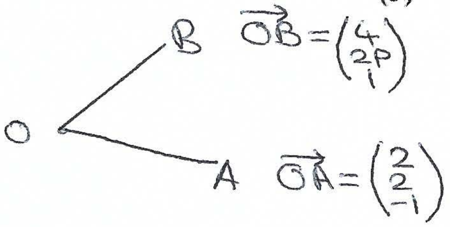
(b) $\overrightarrow{OA} \times \overrightarrow{OB}$ is parallel to $\begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}$

(3)

(c) the area of triangle OAB is $3\sqrt{2}$

(3)

Solution 3a



$$\cos 45^\circ = \frac{4 \;\leftarrow \text{adjacent}}{|\overrightarrow{OB}| \;\leftarrow \text{hypoteneuse}}$$

$$\Rightarrow \frac{\sqrt{2}}{2} = \frac{4}{\sqrt{4^2 + (2p)^2 + 1^2}} = \frac{4}{\sqrt{17 + 4p^2}}$$

$$\Rightarrow \xrightarrow{\text{Rearranging}} \ldots \Rightarrow 8p^2 = 30 \Rightarrow p^2 = \frac{15}{4} \Rightarrow p = \pm\frac{\sqrt{15}}{2}$$

Solution 3b

$$\overrightarrow{OA} \times \overrightarrow{OB} = \lambda \begin{pmatrix} 4 \\ -p \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2p \\ 1 \end{pmatrix} = \begin{pmatrix} 4\lambda \\ -p\lambda \\ 2\lambda \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 4 & 2p & 1 \end{vmatrix} = \begin{pmatrix} 4\lambda \\ -p\lambda \\ 2\lambda \end{pmatrix} \Rightarrow \begin{pmatrix} 2+2p \\ -6 \\ 4p-8 \end{pmatrix} = \begin{pmatrix} 4\lambda \\ -p\lambda \\ 2\lambda \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2+2p &= 4\lambda \quad (1) \\ 6 &= p\lambda \quad (2) \\ 4p-8 &= 2\lambda \quad (3) \end{aligned}\right\}$$

Sub (3) in (1) $\quad 2+2p = 2(4p-8)$

$$\Rightarrow 2+2p = 8p-16$$
$$\Rightarrow 6p = 18$$
$$\Rightarrow p = 3$$

Solution 3c

$$\frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}| = 3\sqrt{2} \Rightarrow \left|\begin{pmatrix} 2+2p \\ -6 \\ 4p-8 \end{pmatrix}\right| = 6\sqrt{2}$$

$$\Rightarrow (2+2p)^2 + (-6)^2 + (4p-8)^2 = (6\sqrt{2})^2 \xrightarrow[\text{factorise}]{\text{rearrange \&}} (5p-4)(p-2) = 0$$

$$\Rightarrow p = \frac{4}{5}, \; p = 2$$

4. The velocity $v\text{ms}^{-1}$, of a raindrop, t seconds after it falls from a cloud, is modelled by the differential equation

$$\frac{dv}{dt} = -0.1v^2 + 10 \qquad t \geqslant 0$$

Initially the raindrop is at rest.

(a) Use two iterations of the approximation formula $\left(\frac{dy}{dx}\right)_n \approx \frac{y_{n+1} - y_n}{h}$ to estimate the velocity of the raindrop 1 second after it falls from the cloud.

(5)

Given that the initial acceleration of the raindrop is found to be smaller than is suggested by the current model,

(b) refine the model by changing the value of one constant.

(1)

Solution 4a

$t_0 = 0, \quad V_0 = 0, \quad \left(\frac{dv}{dt}\right)_0 = 10, \quad h = 0.5$

$V_1 = V_0 + h\left(\frac{dv}{dt}\right)_0 \Rightarrow V_1 = 0 + 0.5 \times 10 = 5$

$\left(\frac{dv}{dt}\right)_1 = -0.1(5)^2 + 10 = 7.5$

$V_2 = V_1 + h\left(\frac{dv}{dt}\right)_1 \Rightarrow V_2 = 5 + 0.5 \times 7.5 = 8.75\text{ms}^{-1}$

Solution 4b

$\frac{dV}{dt} = -0.1V^2 + A, \quad 0 < A < 10$

5. The rectangular hyperbola H has equation $xy = 36$

(a) Use calculus to show that the equation of the tangent to H at the point $P\left(6t, \frac{6}{t}\right)$ is

$$yt^2 + x = 12t$$

(3)

The point $Q\left(12t, \frac{3}{t}\right)$ also lies on H.

(b) Find the equation of the tangent to H at the point Q.

(2)

The tangent at P and the tangent at Q meet at the point R.

(c) Show that as t varies the locus of R is also a rectangular hyperbola.

(4)

Solution 5a

$xy = 36$. Differentiate implicitly: $x\frac{dy}{dx} + y = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$.

At $x = 6t$, $y = \frac{6}{t}$ $\Rightarrow \frac{dy}{dx} = \frac{-6/t}{6t} \Rightarrow \frac{dy}{dx} = -\frac{1}{t^2}$

Point P is $\left(6t, \frac{6}{t}\right)$ and gradient $\frac{dy}{dx} = -\frac{1}{t^2}$

So $y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t) \Rightarrow yt^2 + x = 12t$ ①

Solution 5b

$\frac{dy}{dx} = \frac{-y}{x}$. At $x = 12t$, $y = \frac{3}{t}$ $\Rightarrow \frac{dy}{dx} = \frac{-3/t}{12t} = -\frac{1}{4t^2}$

Point Q is $\left(12t, \frac{3}{t}\right)$, gradient $\frac{dy}{dx} = -\frac{1}{4t^2}$

So $y - 3/t = \left(\frac{-1}{4t^2}\right)(x - 12t) \Rightarrow \ldots \Rightarrow 4yt^2 + x = 24t$ ②

Solution 5c

① $yt^2 + x = 12t \overset{\times 2}{\Rightarrow} 2yt^2 + 2x = 24t$ ②

$\Rightarrow 2yt^2 + 2x = 4yt^2 + x$

$\Rightarrow \boxed{2yt^2 = x}$

② $\Rightarrow 4yt^2 + \boxed{2yt^2} = 24t \Rightarrow y = \boxed{\frac{4}{t}} \Rightarrow \boxed{x = 2\left(\frac{4}{t}\right)t^2 = 8t}$

So $x = 8t$ and $y = \frac{4}{t} \Rightarrow xy = 32$

Hence this is a rectangular hyperbola

6. The points P, Q and R have position vectors $\begin{pmatrix}1\\-2\\4\end{pmatrix}$, $\begin{pmatrix}3\\1\\-5\end{pmatrix}$ and $\begin{pmatrix}2\\0\\3\end{pmatrix}$ respectively.

(a) Determine a vector equation of the plane that passes through the points P, Q and R, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalar parameters.

(2)

(b) Determine the coordinates of the point of intersection of the plane with the x-axis.

(4)

Solution 6a

$$\overrightarrow{OP} = \begin{pmatrix}1\\-2\\4\end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix}3\\1\\-5\end{pmatrix} \quad \overrightarrow{OR} = \begin{pmatrix}2\\0\\3\end{pmatrix}$$

$$\underline{b} = \begin{pmatrix}3\\1\\-5\end{pmatrix} - \begin{pmatrix}1\\-2\\4\end{pmatrix}, \quad \underline{c} = \begin{pmatrix}2\\0\\3\end{pmatrix} - \begin{pmatrix}1\\-2\\4\end{pmatrix}$$

$$\Rightarrow \underline{b} = \begin{pmatrix}2\\3\\-9\end{pmatrix}, \quad \underline{c} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\Rightarrow \underline{r} = \begin{pmatrix}1\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-9\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

Solution 6b

When plane intersects x-axis, $y=0$ and $z=0$.

So $\quad 0 = -2 + 3\lambda + 2\mu \quad (1)$

$\quad 0 = 4 - 9\lambda - \mu \quad (2)$

$3\times(1) + (2) \Rightarrow$

$$\begin{array}{r} -6 + 9\lambda + 6\mu = 0 \\ 4 - 9\lambda - \mu = 0 \\ \hline -2 + 0 + 5\mu = 0 \end{array}$$

$\Rightarrow \boxed{\mu = \frac{2}{5}}$

Substitute $\mu = 2/5$ in (1):

$0 = -2 + 3\lambda + 4/5 \Rightarrow 15\lambda = 10 - 4 = 6$

$\Rightarrow \boxed{\lambda = 2/5}$

$\Rightarrow x = 1 + \boxed{(2/5)}(2) + \boxed{(2/5)}(1) = 11/5 = 2.2$

Point of intersection with plane is: $(2.2, 0, 0)$

7.

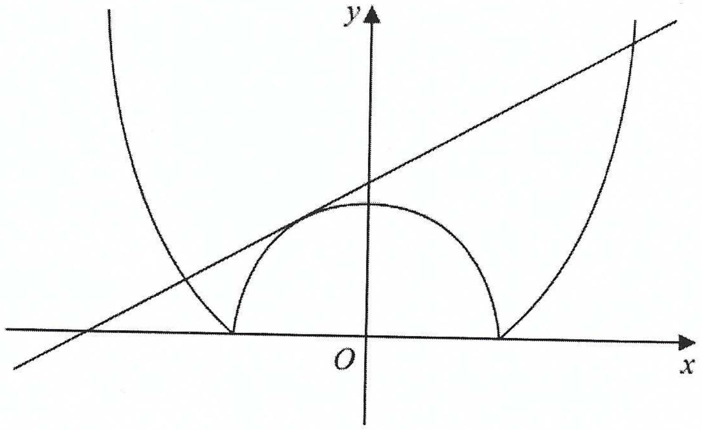


Figure 1

Figure 1 shows a sketch of the curve with equation $y = |x^2 - 8|$ and a sketch of the straight line with equation $y = mx + c$, where m and c are positive constants.

The equation

$$|x^2 - 8| = mx + c$$

has exactly 3 roots, as shown in Figure 1.

(a) Show that

$$m^2 - 4c + 32 = 0$$

(2)

Given that $c = 3m$

(b) determine the value of m and the value of c

(3)

(c) Hence solve

$$|x^2 - 8| \geqslant mx + c$$

(3)

Solution 7a

$x^2 - 8 = -(mx + c) \Rightarrow x^2 + mx + (c-8) = 0$

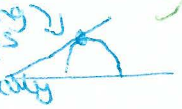
Since this only has one root,

"$b^2 - 4ac$" $= 0$

$\Rightarrow m^2 - 4(c-8) = 0$

$\Rightarrow m^2 - 4c + 32 = 0$

we are looking at this root especially



Solution 7b

$c = 3m$

$$m^2 - 4(3m) + 32 = 0$$

$$\Rightarrow m^2 - 12m + 32 = 0$$

$$\Rightarrow (m-8)(m-4) = 0$$

$$\Rightarrow m = 8,\ m = 4$$

Explanation not required

Using graphing calculator, if $m = 8$, $c = 24$, then there are only 2 intersectors with the graph, but there should be 3.

So, reject $m = 8$

Hence $m = 4 \Rightarrow c = 12$

Solution 7c

Case 1

$$x^2 - 8 = 4x + 12 \quad \Rightarrow x^2 - 4x - 20 = 0$$

$$\Rightarrow x = 2 \pm \sqrt{24}$$

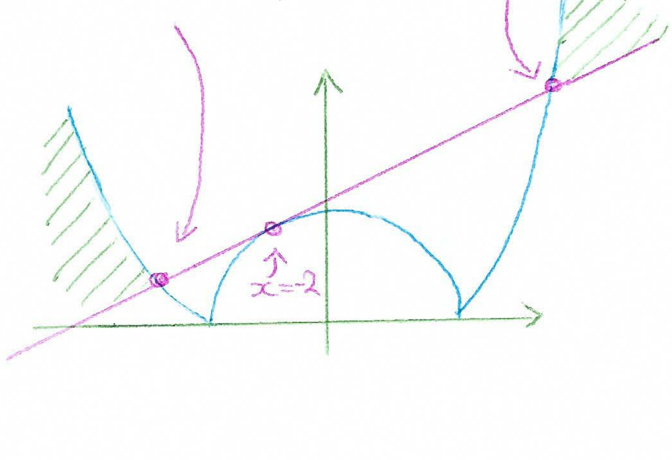
Case 2

$$x^2 - 8 = -(4x+12) \Rightarrow x^2 + 4x + 4 = 0$$

$$\Rightarrow (x+2)^2 = 0$$

$$\Rightarrow x = -2$$

$$\Rightarrow \quad x \leq 2 - 2\sqrt{6},\quad x \geq 2 + 2\sqrt{6},\quad x = -2$$



8. *The Taylor series expansion of* $\mathrm{f}(x)$ *about* $x = a$ *is given by*

$$\mathrm{f}(x) = \mathrm{f}(a) + (x-a)\mathrm{f}'(a) + \frac{(x-a)^2}{2!}\mathrm{f}''(a) + \ldots + \frac{(x-a)^r}{r!}\mathrm{f}^{(r)}(a) + \ldots$$

(i) (a) Use differentiation to determine the Taylor series expansion of $\ln x$, in ascending powers of $(x-1)$, up to and including the term in $(x-1)^2$

(4)

(b) Hence prove that

$$\lim_{x \to 1}\left(\frac{\ln x}{x-1}\right) = 1$$

(2)

(ii) Use L'Hospital's rule to determine

$$\lim_{x \to 0}\left(\frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)}\right)$$

(Solutions relying entirely on calculator technology are not acceptable.)

(4)

Solution 8a

Let $f(x) = \ln(x)$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f(x) \approx f(1) + (x-1)f'(1) + (x-1)^2 \frac{f''(1)}{2!}$$

$$= 0 + (x-1)\frac{1}{1} + \frac{(x-1)^2}{2!} \times \left(-\frac{1}{1^2}\right)$$

$$= (x-1) - \frac{(x-1)^2}{2}$$

Solution 8ib

$$\lim_{x\to 1} \frac{\ln x}{x-1} \approx \lim_{x\to 1} \left(\frac{(x-1) - \frac{(x-1)^2}{2}}{x-1} \right)$$

$$= \lim_{x\to 1} \left(\frac{(x-1)\left(1-\frac{1}{2}(x-1)\right)}{x-1} \right)$$

$$= \lim_{x\to 1} \left[1 - \tfrac{1}{2}(x-1)\right]$$

$$= 1 - \frac{1}{2}(0)$$

$$= 1$$

Solution ii

$$L = \lim_{x\to 0} \frac{1}{(x+3)\tan 6x \operatorname{cosec} 2x}$$

$$= \lim_{x\to 0} \frac{(\cos 6x)(\sin 2x)}{(x+3)\sin 6x}$$

At $x=0$:

$$\frac{\cos 0 \times \sin 0}{(0+3)\sin 0} = \frac{0}{0}$$

$\therefore$ L'Hôpital's rule is applicable

$$L = \lim_{x\to 0} \frac{\cos 6x \times 2\cos 2x - 6\sin 6x \sin 2x}{6(x+3)\cos 6x + \sin 6x}$$

$$= \frac{\cos 0 \times 2\cos 0 - 6\sin 0 \times \sin 0}{6(0+3)\cos 0 + \sin 0}$$

$$= \frac{2}{18}$$

$$= \frac{1}{9}$$

9. A particle P moves along a straight line.

At time t minutes, the displacement, x metres, of P from a fixed point O on the line is modelled by the differential equation

$$t^2 \frac{d^2 x}{dt^2} - 2t\frac{dx}{dt} + 2x + 16t^2 x = 4t^3 \sin 2t \qquad \text{(I)}$$

(a) Show that the transformation $x = ty$ transforms equation (I) into the equation

$$\frac{d^2 y}{dt^2} + 16y = 4\sin 2t$$

(5)

(b) Hence find a general solution for the displacement of P from O at time t minutes.

(8)

Solution 9a

$$x = ty \Rightarrow \frac{dx}{dt} = y + t\frac{dy}{dt}$$ product rule and implicit differentiation

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dy}{dt} + t\frac{d^2y}{dt^2} + \frac{dy}{dt}$$

$$= 2\frac{dy}{dt} + t\frac{d^2y}{dt^2}$$

Substitute in (I):

$$t^2\Big(\overbrace{2\frac{dy}{dt} + t\frac{d^2y}{dt^2}}^{\frac{d^2x}{dt^2}}\Big) - 2t\Big(\overbrace{y + t\frac{dy}{dt}}^{\frac{dx}{dt}}\Big) + 2\overbrace{(ty)}^{x} + 16t^2\overbrace{(ty)}^{x} = 4t^3\sin 2t$$

$$\Rightarrow \ldots \Rightarrow t^3\frac{d^2y}{dt^2} + 16t^3 y = 4t^3 \sin 2t$$

$$\Rightarrow \frac{d^2y}{dt^2} + 16y = 4\sin 2t \quad (*)$$

Solution 9b

AE: $\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i \Rightarrow y = A\cos 4t + B\sin 4t$

PI: $y = \lambda \sin 2t + \mu \cos 2t \Rightarrow \frac{dy}{dt} = 2\lambda\cos 2t - 2\mu \sin 2t$

$$\Rightarrow \frac{d^2y}{dt^2} = -4\lambda \sin 2t - 4\mu\cos 2t$$

Substitute in (*) and simplify ...

Solution 9b continued

$$-4\lambda\sin 2t - 4\mu\cos 2t + 16(\lambda\sin 2t + \mu\cos 2t) = 4\sin 2t$$

$$\Rightarrow -\lambda\sin 2t - \mu\cos 2t + 4\lambda\sin 2t + 4\mu\cos 2t = \sin 2t$$

$$\Rightarrow \boxed{3\lambda\sin 2t} + \boxed{3\mu\cos 2t} = \boxed{\sin 2t} + \boxed{0\cos 2t}$$

$$\Rightarrow 3\lambda = 1,\ \mu = 0 \quad \Rightarrow \lambda = \tfrac{1}{3},\ \mu = 0$$

$$\Rightarrow y = A\cos 4t + B\sin 4t + \tfrac{1}{3}\sin 2t$$

But $x = yt$.

So

$$x = \left(A\cos 4t + B\sin 4t + \tfrac{1}{3}\sin 2t\right)t$$