1. [*In this question position vectors are given relative to a fixed origin O*]

At time t seconds, where $t \geqslant 0$, a particle, P, moves so that its velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$ is given by

$$\mathbf{v} = 6t\mathbf{i} - 5t^{\frac{3}{2}}\mathbf{j}$$

When $t = 0$, the position vector of P is $(-20\mathbf{i} + 20\mathbf{j})\,\mathrm{m}$.

(a) Find the acceleration of P when $t = 4$

(3)

(b) Find the position vector of P when $t = 4$

(3)

Solution 1a

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(6t\underline{i} - 5t^{3/2}\underline{j}\right)$$

$$= 6\underline{i} - 5\left(\frac{3}{2}\right)t^{3/2-1}\underline{j}$$

$$= 6\underline{i} - \frac{15}{2}t^{1/2}\underline{j}$$

When $t=4$, $\underline{a} = 6\underline{i} - \frac{15}{2}(4)^{1/2}\underline{j} = 6\underline{i} - \frac{15}{2}(2)\underline{j}$

$$\Rightarrow \quad \underline{a} = 6\underline{i} - 15\underline{j}$$

Solution 1b

$$\underline{s} = \int \underline{v}\,dt = \int 6t\underline{i} - 5t^{3/2}\underline{j}\,dt = \frac{6t^2}{2}\underline{i} - \frac{5t^{3/2+1}}{3/2+1} + \underline{s}_0$$

$$= 3t^2\underline{i} - 2t^{5/2}\underline{j} + \underline{s}_0$$

When $\boxed{t=0}$, position vector of P is $(-20\underline{i}+20\underline{j})\mathrm{m}$

$$\therefore \quad 3(0)^2\underline{i} - 2(0)^{5/2}\underline{j} + \underline{s}_0 = -20\underline{i} + 20\underline{j}$$

$$\Rightarrow \quad \underline{s}_0 = -20\underline{i} + 20\underline{j}$$

Hence $\underline{s} = 3t^2\underline{i} - 2t^{5/2}\underline{j} - 20\underline{i} + 20\underline{j}$

$$\Rightarrow \quad \underline{s} = (3\boxed{t^2} - 20)\underline{i} + (20 - 2\boxed{t^{5/2}})\underline{j}$$

When $\boxed{t=4}$: $\underline{s} = (3\boxed{(4)}^2 - 20)\underline{i} + (20 - 2\boxed{(4^{5/2})})\underline{j}$

$$\Rightarrow \quad \underline{s} = 28\underline{i} - 44\underline{j}$$

2. A particle, P, moves with constant acceleration $(2\mathbf{i} - 3\mathbf{j})\,\mathrm{m\,s^{-2}}$

At time $t = 0$, the particle is at the point A and is moving with velocity $(-\mathbf{i} + 4\mathbf{j})\,\mathrm{m\,s^{-1}}$

At time $t = T$ seconds, P is moving in the direction of vector $(3\mathbf{i} - 4\mathbf{j})$

λ

(a) Find the value of T.

(4)

At time $t = 4$ seconds, P is at the point B.

(b) Find the distance AB.

(4)

(Solution overleaf)

Solution 2a

$\underline{v} = \underline{u} + \underline{a}t$

From question, $\boxed{\underline{a}} = \boxed{(2\underline{i} - 3\underline{j})} \text{ms}^{-2}$

$\boxed{\underline{u}} = \boxed{(-\underline{i} + 4\underline{j})} \text{ms}^{-1}$

At time $\boxed{t = T}$, $\boxed{\underline{v}} = \boxed{(3\underline{i} - 4\underline{j})\lambda} \text{ms}^{-1}$ ← unknown λ

Hence substituting in $\underline{v} = \boxed{\underline{u}} + \boxed{\underline{a}}\boxed{t}$ gives:

$$\boxed{(3\underline{i} - 4\underline{j})\lambda} = \boxed{(-\underline{i} + 4\underline{j})} + \boxed{(2\underline{i} - 3\underline{j})}\boxed{T}$$

$$\Rightarrow \boxed{\begin{pmatrix} 3 \\ -4 \end{pmatrix}\lambda} = \boxed{\begin{pmatrix} -1 \\ 4 \end{pmatrix}} + \boxed{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}\boxed{T}$$

$\Rightarrow 3\lambda = -1 + 2T$ ① $\times 3 \Rightarrow 9\lambda = -3 + 6T$ +

$-4\lambda = 4 - 3T$ ② $\times 2 \Rightarrow -8\lambda = 8 - 6T$ +

$\Rightarrow \lambda = 5$

Substitute in ①

$3(5) = -1 + 2T$

$\Rightarrow 15 + 1 = 2T$

$\Rightarrow T = 8$

Solution 2b

$\underline{s} = \boxed{\underline{u}}t + \frac{1}{2}\boxed{\underline{a}}t^2$

$\underline{s}_{\overrightarrow{AB}} = \boxed{(-\underline{i} + 4\underline{j})}t + \frac{1}{2}\boxed{(2\underline{i} - 3\underline{j})}t^2$

$\Rightarrow \underline{s}_{\overrightarrow{AB}} = (-\underline{i} + 4\underline{j})t + (\underline{i} - \frac{3}{2}\underline{j})t^2$

When $t = 4$: $\underline{s}_{\overrightarrow{AB}} = (-\underline{i} + 4\underline{j})(4) + (\underline{i} - \frac{3}{2}\underline{j})(4)$

$= 12\underline{i} - 8\underline{j}$

$\Rightarrow |AB| = \sqrt{12^2 + 8^2} = \sqrt{208} = 14.42$

3.

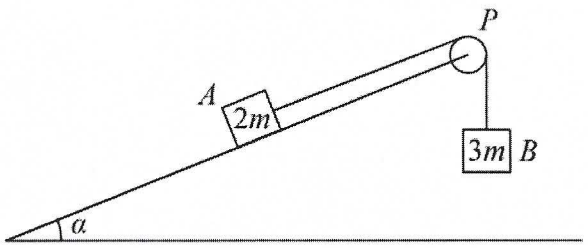


Figure 1

Two blocks, A and B, of masses $2m$ and $3m$ respectively, are attached to the ends of a light string.

Initially A is held at rest on a fixed rough plane.

The plane is inclined at angle α to the horizontal ground, where $\tan\alpha = \frac{5}{12}$

The string passes over a small smooth pulley, P, fixed at the top of the plane.

The part of the string from A to P is parallel to a line of greatest slope of the plane. Block B hangs freely below P, as shown in Figure 1.

The coefficient of friction between A and the plane is $\frac{2}{3}$

The blocks are released from rest with the string taut and A moves up the plane.

The tension in the string immediately after the blocks are released is T.

The blocks are modelled as particles and the string is modelled as being inextensible.

(a) Show that $T = \frac{12mg}{5}$

(8)

After B reaches the ground, A continues to move up the plane until it comes to rest before reaching P.

(b) Determine whether A will remain at rest, carefully justifying your answer.

(2)

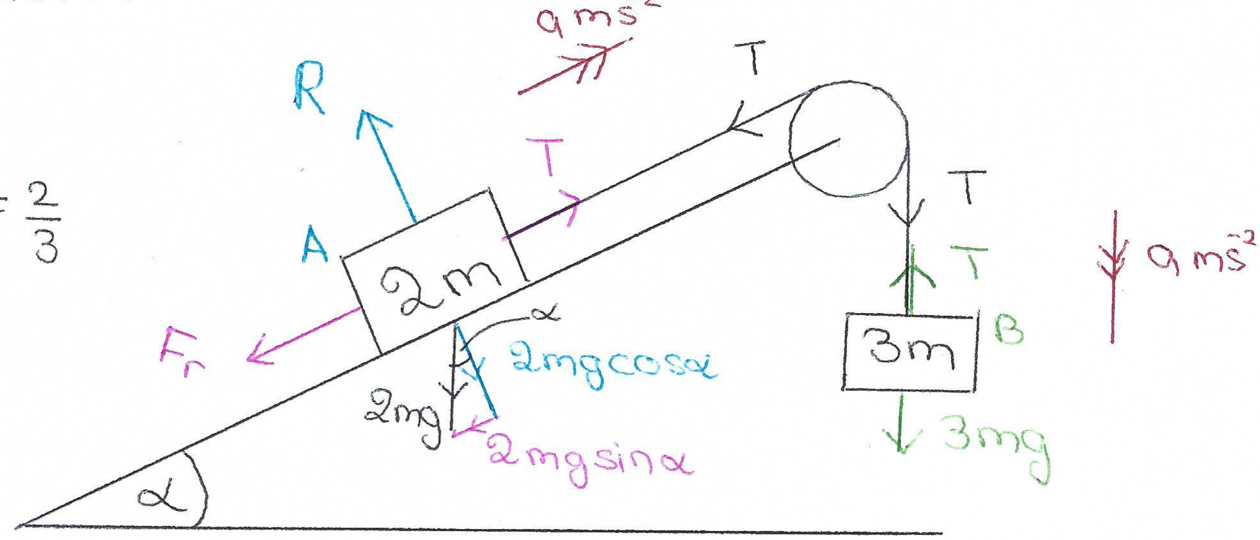
(c) Suggest two refinements to the model that would make it more realistic.

(2)

Solution 3a, b, c overleaf

Solution 3a

$\mu = \frac{2}{3}$



$R_A(\uparrow)$:

$R = 2mg\cos\alpha$

$= 2mg\left(\frac{12}{13}\right)$

$= \frac{24}{13}mg$

$R_B(\uparrow)$: $\underline{F} = m\underline{a}$

$3mg - T = 3m\underline{a}$ ①

$R_A(\nearrow)$: $\underline{F} = m\underline{a}$

$T - F_r - 2mg\sin\alpha = 2m\underline{a}$

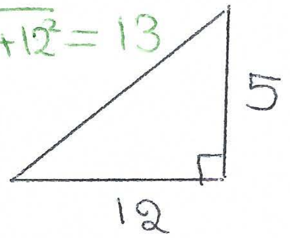
$\Rightarrow T - \mu R - 2mg\sin\alpha = 2m\underline{a}$

$\Rightarrow T - \frac{2}{3}\left(\frac{24}{13}mg\right) - 2mg\left(\frac{5}{13}\right) = 2m\underline{a}$

$\Rightarrow T - \frac{16}{13}mg - \frac{10}{13}mg = 2ma$

$\Rightarrow \boxed{T - 2mg = 2m\underline{a}}$ ②

$\sqrt{5^2+12^2} = 13$



$\tan\alpha = \frac{5}{12}$

$\Rightarrow \sin\alpha = \frac{5}{13}$

$\Rightarrow \cos\alpha = \frac{12}{13}$

① + ②

$mg = 5m\underline{a}$

$\Rightarrow \boxed{\underline{a} = \frac{1}{5}g}$

Substitute in ②

$\boxed{T - 2mg = 2m\left(\boxed{\frac{1}{5}g}\right)}$

$\Rightarrow T = 2mg + \frac{2}{5}mg$

$\Rightarrow T = \frac{12}{5}mg$

as required

Solution 3b

$$\boxed{2mg\sin\alpha} = 2mg \times \frac{5}{13}$$
$$= \boxed{\frac{10}{13}mg}$$

$$\Rightarrow \boxed{F_r} = \mu R$$
$$= \frac{2}{3} \times \frac{24}{13}mg = \boxed{\frac{16}{13}mg}$$

Now $\boxed{\frac{16}{13}mg} > \boxed{\frac{10}{13}mg}$

$$\Rightarrow \quad F_r > 2mg\sin\alpha$$

Yes, A will stay at rest, as the string has become slack at this point, so the only forces acting on A are its weight, and friction.

But the force of weight parallel to the slope ($2mg\sin\alpha$) is less than the maximum friction (F_r).

So A will stay at rest

Solution 3c

Two refinements to make model more realistic:

- not modelling block as a particle
- not modelling the string as inextensible.

4.

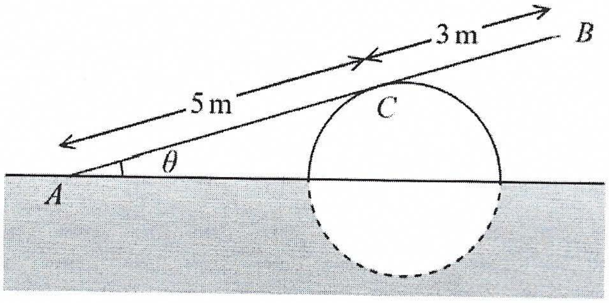


Figure 2

A ramp, AB, of length 8 m and mass 20 kg, rests in equilibrium with the end A on rough horizontal ground.

The ramp rests on a smooth solid cylindrical drum which is partly under the ground. The drum is fixed with its axis at the same horizontal level as A.

The point of contact between the ramp and the drum is C, where $AC = 5$ m, as shown in Figure 2.

The ramp is resting in a vertical plane which is perpendicular to the axis of the drum, at an angle θ to the horizontal, where $\tan\theta = \frac{7}{24}$

The ramp is modelled as a uniform rod.

(a) Explain why the reaction from the drum on the ramp at point C acts in a direction which is perpendicular to the ramp.

(1)

(b) Find the magnitude of the resultant force acting on the ramp at A.

(9)

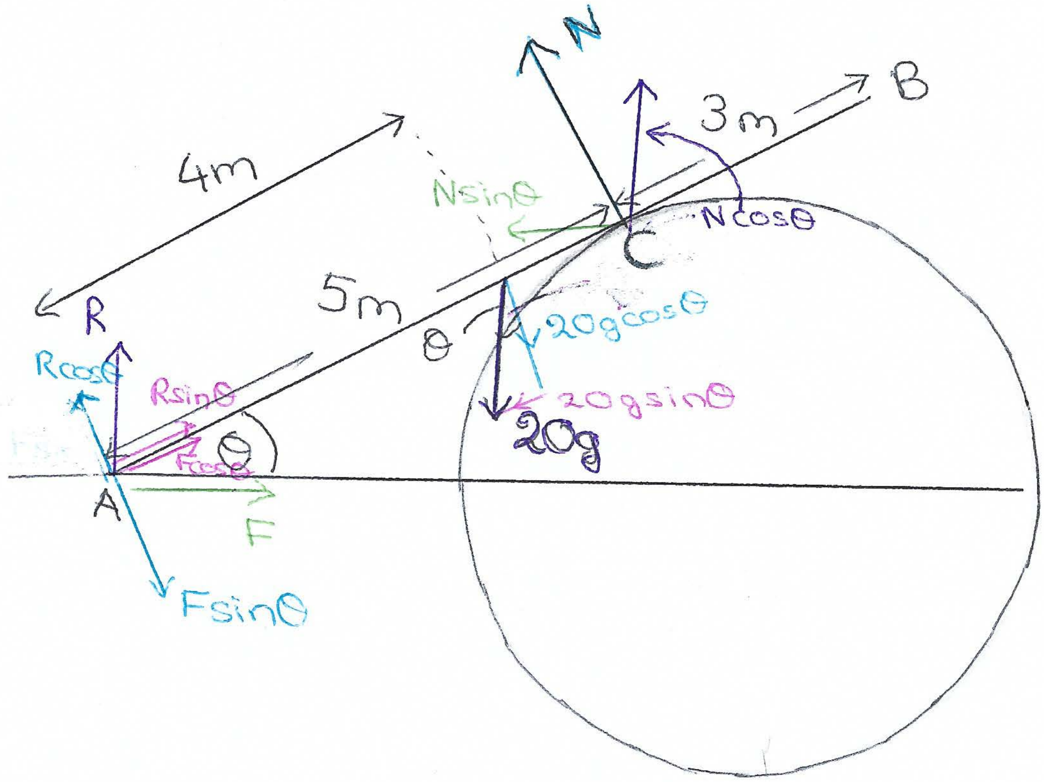
The ramp is still in equilibrium in the position shown in Figure 2 but the ramp is not now modelled as being uniform.

Given that the centre of mass of the ramp is assumed to be closer to A than to B,

(c) state how this would affect the magnitude of the normal reaction between the ramp and the drum at C.

(1)

Solution 4a



The drum is smooth (or there is no friction). Therefore the reaction is perpendicular to the ramp.

Solution 4b

R (↗): $F\cos\theta + R\sin\theta = 20g\sin\theta$

R (↖): $N + R\cos\theta = 20g\cos\theta + F\sin\theta$

R (↑): $R + N\cos\theta = 20g$

R (→): $F = N\sin\theta$

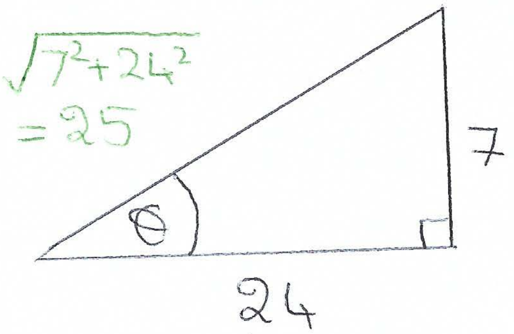
M(A): $4(20g\cos\theta) = 5N$

M(B): $3N + 8R\cos\theta = 4(20g\cos\theta) + 8F\sin\theta$

M(C): $5R\cos\theta = 5F\sin\theta + 20g\cos\theta$

M(G): $4R\cos\theta = 4F\sin\theta + N$

Solution 4b continued



$$\tan(\theta)=\frac{7}{24}$$

$$\Rightarrow \cos\theta=\frac{24}{25}$$

$$\Rightarrow \sin\theta=\frac{7}{25}$$

Substituting in equations:

$$(\nearrow):\ F\left(\frac{24}{25}\right)+R\left(\frac{7}{25}\right)=20g\left(\frac{7}{25}\right)$$

$$(\nwarrow):\ N+R\left(\frac{24}{25}\right)=20g\left(\frac{24}{25}\right)+F\left(\frac{7}{25}\right)$$

$$(\uparrow):\ R+N\left(\frac{24}{25}\right)=20g$$

$$(\rightarrow):\ F=N\left(\frac{7}{25}\right)$$

$$M(A):\ 16g\left(\frac{24}{25}\right)=N$$

$$M(B):\ 3N+8R\left(\frac{24}{25}\right)=80g\left(\frac{24}{25}\right)+8F\left(\frac{7}{25}\right)$$

$$M(C):\ 5R\left(\frac{24}{25}\right)=5F\left(\frac{7}{25}\right)+20g\left(\frac{24}{25}\right)$$

$$M(G):\ 4R\left(\frac{24}{25}\right)=4F\left(\frac{7}{25}\right)+N$$

Solution 4b continued

① $24F + 7R = 140g$

② $25\boxed{N} + 24\boxed{R} = 480g + 7F$

③ $25\boxed{R} + 24\boxed{N} = 500g$

④ $25F = 7N$

⑤ $\frac{384g}{25} = N$

⑥ $75N + 192R = 1920g + 56F$

⑦ $120R = 35F + 480g$

⑧ $96R = 28F + 25N$

Equation ⑤ $\Rightarrow \boxed{N = 15.36g} \approx 150.528 \quad (g = 9.8ms^{-2})$

Substitute N in ③ $\Rightarrow 25R + 24 \times \boxed{15.36g} = 500g$

$\Rightarrow 25R = 131.36g$

$\Rightarrow \boxed{R = 5.2544g} \approx 51.49$

Substitute N and R in ②

$\Rightarrow 25(\boxed{15.36g}) + 24(\boxed{5.2544g}) = 480g + 7F$

$\Rightarrow 7F = 30.1056g$

$\Rightarrow F = 4.3008g \approx 42.14784 \Rightarrow |Force| = \sqrt{R^2 + F^2} = 66.5$

Solution 4c

The magnitude of the normal reaction (at C) will decrease.

5.

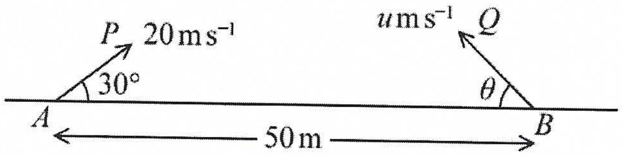


Figure 3

The points A and B lie 50 m apart on horizontal ground.

At time $t = 0$ two small balls, P and Q, are projected in the vertical plane containing AB.

Ball P is projected from A with speed $20\,\text{m s}^{-1}$ at 30° to AB.

Ball Q is projected from B with speed $u\,\text{m s}^{-1}$ at angle θ to BA, as shown in Figure 3.

At time $t = 2$ seconds, P and Q collide.

Until they collide, the balls are modelled as particles moving freely under gravity.

(a) Find the velocity of P at the instant before it collides with Q.

(6)

(b) Find

(i) the size of angle θ,

(ii) the value of u.

(6)

(c) State one limitation of the model, other than air resistance, that could affect the accuracy of your answers.

(1)

Solution 5a

$R_P(\rightarrow)$: $s = ut$

$x = (20\cos 30°)t$

$\Rightarrow x = (20\cos 30°)(2)$

$\Rightarrow \boxed{x = 20\sqrt{3}}$

$R_P(\uparrow)$: $s = ut + \frac{1}{2}at^2$

$h = (20\sin 30°)(2) - \frac{1}{2}g(2)^2$

$= 20 - 2g$

$R_P(\nearrow)$: $v_y = u + at$

$= 20\sin 30 - g \times 2$

$= 10 - 2g$

$R_Q(\rightarrow)$: $s = ut$

$50 - x = (u\cos\theta)(t)$

$\Rightarrow 50 - x = (u\cos\theta)(2)$

$\Rightarrow 50 - \boxed{x} = 2u\cos\theta$

$\Rightarrow 50 - \boxed{20\sqrt{3}} = 2u\cos\theta$

$\Rightarrow \boxed{u\cos\theta = 25 - 10\sqrt{3}}$ (*)

$R_Q(\uparrow)$: $s = ut + \frac{1}{2}at^2$

$20 - 2g = (u\sin\theta)(2) - \frac{1}{2}g(2)^2$

$= 2u\sin\theta - 2g$

$\Rightarrow 20 = 2u\sin\theta \Rightarrow \boxed{10 = u\sin\theta}$ (**)

$R_P(\rightarrow)$: $v_x = \frac{s}{t} = \frac{20\sqrt{3}}{2} = 10\sqrt{3}$

$\Rightarrow |\underline{v}| = \sqrt{v_x^2 + v_y^2} = 19.8\,\text{ms}^{-1}$, $\alpha = \tan^{-1}\left(\frac{10-2g}{10\sqrt{3}}\right)$

$29°$

Solution 5bi

$$\frac{\text{Equation } (**)}{\text{Equation } (*)} \Rightarrow \tan\theta = \frac{10}{25-10\sqrt{3}}$$

$$\Rightarrow \theta = 52.5° \ (3sf)$$

Solution 5bii

Equation $\Rightarrow$

$$10 = u\sin\theta$$

$$10 = u\sin(52.47...)$$

$$\Rightarrow u = 12.6 \ (3sf)$$

Solution 5c

A limitation of the model is that the balls are modelled as particles.