1. Given that

$$z_1 = 3\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$
$$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) - \mathrm{i}\sin\left(\frac{\pi}{12}\right)\right)$$

(a) write down the exact value of

(i) $|z_1 z_2|$

(ii) $\arg(z_1 z_2)$

(2)

Given that $w = z_1 z_2$ and that $\arg(w^n) = 0$, where $n \in \mathbb{Z}^+$

(b) determine

(i) the smallest positive value of n

(ii) the corresponding value of $|w^n|$

(3)

Solution 1ai and 1aii

$z_1 = 3e^{i\pi/3}$, $z_2 = \sqrt{2}\,e^{(-i\pi/12)}$

Hence $z_1 z_2 = \left(3e^{i\pi/3}\right)\left(\sqrt{2}\,e^{(-i\pi/12)}\right)$

$= 3\sqrt{2}\,e^{i\pi/3 - i\pi/12} = 3\sqrt{2}\,e^{+3i\pi/12}$

$= 3\sqrt{2}\,e^{+i\pi/4} = 3\sqrt{2}\left(\cos\left(+\pi/4\right) + i\sin\left(+\pi/4\right)\right)$

So

$|z_1 z_2| = 3\sqrt{2}$

and $\arg(z_1 z_2) = \frac{\pi}{4}$

Solution 1bi and 1bii

$w = z_1 z_2$ $\arg(w^n) = 0$

$w^n = \left(\left(3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)\right)^n = \left(3\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

Find smallest n for which $\sin\frac{n\pi}{4} = 0$

$\Rightarrow \frac{n\pi}{4} = 2\pi \Rightarrow n = 8$

$\Rightarrow |w^n| = |w^8| = (3\sqrt{2})^8 = 3^8(\sqrt{2})^8 = 104976$

2.

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix **A** represents the linear transformation M.

Prove that, for the linear transformation M, there are no invariant lines.

(5)

Solution 2 Try proof by contradiction

Suppose that for ~~the~~ linear transformation M, there is an invariant line.

So

$$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix}$$

$\Rightarrow 4x - 2mx - 2c = X$ ①

$5x + 3mx + 3c = mX + c$ ②

Substitute ① in ② to give

$5x + 3mx + 3c = m(4x - 2mx - 2c) + c$

$\Rightarrow 5x + 3mx + 3c = 4mx - 2m^2x - 2cm + c$

$\Rightarrow (5 - m + 2m^2)x + 2c + 2cm = 0$

So $2m^2 - m + 5 = 0$ ③

and $2c + 2cm = 0$ ④ $\Rightarrow m = -1$

③ $\Rightarrow m = \dfrac{1 \pm \sqrt{1 - 4(2)(5)}}{4} = \dfrac{1 \pm \sqrt{-39}}{4}$ $\Rightarrow$ no solutions so no invariant lines here

④ $\Rightarrow m = -1$

Plugging $m = -1$ into ③ gives:

$\left.\begin{array}{l} \text{LHS} = 2(-1) - (-1) + 5 = 4 \\ \text{RHS} = 0 \end{array}\right\}$ $\Rightarrow$ LHS $\neq$ RHS $\Rightarrow$ no invariant lines here.

3. $\qquad \mathrm{f}(x) = \arcsin x \qquad -1 \leqslant x \leqslant 1$

(a) Determine the first two non-zero terms, in ascending powers of x, of the Maclaurin series for $\mathrm{f}(x)$, giving each coefficient in its simplest form.

(4)

(b) Substitute $x = \frac{1}{2}$ into the answer to part (a) and hence find an approximate value for π

Give your answer in the form $\frac{p}{q}$ where p and q are integers to be determined.

(2)

Solution 3a

$f(x) = \arcsin x$ (*) $\Rightarrow f'(x) = (1-x^2)^{-1/2}$ (**)

If $y = (1-x^2)^{-1/2} \Rightarrow y = u^{-1/2}$ for $u = 1-x^2$

using chainrule

$\Rightarrow \boxed{\frac{dy}{du}} = \boxed{-\frac{1}{2}u^{-3/2}}, \boxed{\frac{du}{dx}} = \boxed{-2x}$

$\Rightarrow \boxed{\frac{dy}{du}} \times \boxed{\frac{du}{dx}} = \boxed{-\frac{1}{2}u^{-3/2}}\boxed{(-2x)}$

$\Rightarrow \boxed{\frac{dy}{dx}} = \boxed{2x}\boxed{\frac{(1-x^2)}{2}^{-3/2}}$

$\Rightarrow f''(x) = x(1-x^2)^{-3/2}$ (***)

$\Rightarrow f''(x) = \frac{\boxed{x}}{\boxed{(1-x^2)^{+3/2}}}$

Let $\boxed{v = (1-x^2)^{3/2}}$, $u = \boxed{x}$

$\Rightarrow \frac{dv}{dx} = (-2x)(1-x^2)^{1/2}(3/2) \qquad \frac{du}{dx} = 1$

By quotient rule,

$f'''(x) = \frac{(1-x^2)^{3/2}(1) - (x)(-3x)(1-x^2)^{1/2}}{(1-x^2)^3}$

$\Rightarrow f'''(x) = \frac{1}{(1-x^2)^{3/2}} + \frac{3x^2}{(1-x^2)^{5/2}}$ (****)

(*) $\Rightarrow \boxed{f(0)} = \arcsin 0 = \boxed{0}$

(**) $\Rightarrow \boxed{f'(0)} = (1-0^2)^{-1/2} = \boxed{1}$

(***) $\Rightarrow \boxed{f''(0)} = 0(1-0^2)^{-3/2} = \boxed{0}$

(****) $\Rightarrow \boxed{f'''(0)} = \frac{1}{(1-0^2)^{3/2}} + \frac{3 \times 0^2}{(1-0^2)^{5/2}} = \boxed{1}$

$\Rightarrow f(x) = \boxed{f(0)} + \boxed{f'(0)}x + \boxed{f''(0)}\frac{x^2}{2!} + \boxed{f'''(0)}\frac{x^3}{3!} + \ldots$

$\Rightarrow f(x) = \boxed{0} + 1x + \boxed{0} + 1\frac{x^3}{6} \Rightarrow f(x) = x + \frac{x^3}{6}$

Solution 3b

$\boxed{f(x) = \arcsin x}$ and from 3a, $\boxed{f(x) = x + \frac{x^3}{6}}$

If $\boxed{x = \frac{1}{2}}$, then $\boxed{f(x) = \arcsin\left(\boxed{\tfrac{1}{2}}\right)} = \frac{\pi}{6}$

and $\boxed{f(x) = \left(\boxed{\tfrac{1}{2}}\right) + \frac{\left(\boxed{\tfrac{1}{2}}\right)^3}{6}} = \frac{24+1}{48} = \frac{25}{48}$

$$\Rightarrow \frac{\pi}{6} = \frac{25}{48} \Rightarrow \pi = \frac{25}{8}$$

4. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3, \quad \sum_{r=1}^{n} r^2 \quad \text{and} \quad \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2-1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

Solution 4a

$$\sum_{r=1}^{n}(2r-1)^3 = \sum_{r=1}^{n}(8r^3-12r^2+6r-1)$$

$$= 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 8\frac{n^2}{4}(n+1)^2 - 12\frac{n}{6}(n+1)(2n+1) + 6\frac{n}{2}(n+1) - n$$

$$= 2n^2(n^2+2n+1) - 2n(2n^2+3n+1) + 3n^2+3n - n$$

$$= 2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$$

$$= 2n^4 \quad -n^2 = n^2(2n^2-1)$$

Solution 4b

$$\sum_{r=n+1}^{n+10}(2r-1)^3 = \sum_{r=1}^{n+10}(2r-1)^3 - \sum_{r=1}^{n}(2r-1)^3$$

$$= (n+10)^2(2(n+10)^2-1) - n^2(2n^2-1) = 99800$$

$$= (n^2+20n+100)(2n^2+40n+199) - 2n^4+n^2 = 99800$$

$$\Rightarrow 2n^4+40n^3+199n^2+40n^3+800n^2+3980n+200n^2$$
$$+4000n+19900-2n^4+n^2 = 99800$$

$$\Rightarrow 80n^3+1200n^2+7980n-79900 = 0$$

$$\Rightarrow 8n^3+120n^2+798n-7990 = 0$$

$$\Rightarrow n=5 \Rightarrow \text{Smallest number is } 2(n+1)-1 = 11$$

5. The curve C has equation

$$y = \arccos\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2$$

(a) Show that C has no stationary points.

(3)

The normal to C, at the point where $x = 1$, crosses the x-axis at the point A and crosses the y-axis at the point B.

Given that O is the origin,

(b) show that the area of the triangle OAB is $\frac{1}{54}\left(p\sqrt{3} + q\pi + r\sqrt{3}\pi^2\right)$ where p, q and r are integers to be determined.

(5)

Solution 5a

$y = \arccos\left(\frac{1}{2}x\right) \Rightarrow \cos y = \frac{1}{2}x$

$\Rightarrow x = 2\cos y$

$\Rightarrow \frac{dx}{dy} = -2\sin y$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{2\sin y}$

Since $\frac{dy}{dx} \neq 0$, the curve C has no stationary points.

Solution 5b

At $x=1$, (Tangent at $x=1$) $\times$ (Normal at $x=1$) $= -1$

Now Tangent at $x=1$ is $\frac{dy}{dx} = \frac{-1}{2\sin(\arccos\frac{1}{2})} = \frac{-1}{\sqrt{3}}$

$\Rightarrow$ Normal at $x=1$ has gradient $\sqrt{3}$

At $x=1$, $y = \frac{\pi}{3}$.

So of normal is $y - \frac{\pi}{3} = \sqrt{3}(x-1)$

To find points A and B:

when $x=0$, $y = \frac{\pi}{3} - \sqrt{3}$ ← negative so switch order for length

when $y=0$, $x = 1 - \frac{\pi}{3\sqrt{3}}$

Area $= -\frac{1}{2}\left(\frac{\pi}{3} - \sqrt{3}\right)\left(1 - \frac{\pi}{3\sqrt{3}}\right) = \frac{1}{54}\left(27\sqrt{3} - 18\pi + \sqrt{3}\pi^2\right)$

$p=27$, $q=-18$, $r=1$

6. The curve C has equation

$$r = a(p + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where a and p are positive constants and $p > 2$

There are exactly four points on C where the tangent is perpendicular to the initial line.

(a) Show that the range of possible values for p is

$$2 < p < 4$$

(5)

(b) Sketch the curve with equation

$$r = a(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi \qquad \text{where } a > 0$$

(1)

John digs a hole in his garden in order to make a pond.

The pond has a uniform horizontal cross section that is modelled by the curve with equation

$$r = 20(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where r is measured in centimetres.

The depth of the pond is 90 centimetres.

Water flows through a hosepipe into the pond at a rate of 50 litres per minute.

Given that the pond is initially empty,

(c) determine how long it will take to completely fill the pond with water using the hosepipe, according to the model. Give your answer to the nearest minute.

(7)

(d) State a limitation of the model.

(1)

Solution 6a

$r = p + 2\cos\theta$

$x = \boxed{r}\cos\theta = a\boxed{(p + 2\cos\theta)}\cos\theta = ap\cos\theta + 2a\cos^2\theta$

$\Rightarrow x = a(p\cos\theta + \cos 2\theta + 1)$

$\Rightarrow \dfrac{dx}{d\theta} = -ap\sin\theta - 2a\sin 2\theta$

$\Rightarrow \dfrac{dx}{d\theta} = -ap\sin\theta - 4a\sin\theta\cos\theta = -\sin\theta(ap + 4a\cos\theta)$

$\Rightarrow$ Either $\sin\theta = 0$ or $\cos\theta = -\dfrac{p}{4}$

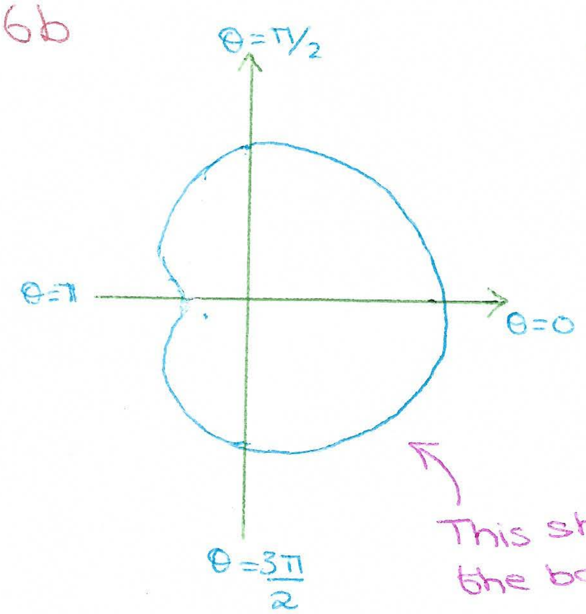
$\sin\theta = 0 \Rightarrow 2$ solutions (tangents perpendicular to initial line) eg $\theta = 0, \pi$

$\therefore 2$ solutions to $\cos\theta = -\dfrac{p}{4}$ are required

$-\dfrac{p}{4} > -1 \Rightarrow p < 4$ as p is a positive constant

$\Rightarrow 2 < p < 4$

Solution 6b



Important values:

$\theta = 0 \Rightarrow r = a(2+p)$

$\theta = \frac{\pi}{2} \Rightarrow r = ap$

$\theta = \pi \Rightarrow r = a(p-2)$

$\theta = \frac{3\pi}{2} \Rightarrow r = ap$

This shape represents the base of the pond

Volume = Area of base x Height

Solution 6c

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^{2\pi} 20^2(3+2\cos\theta)^2\,d\theta$$

$$= 200\int_0^{2\pi} 9 + 12\cos\theta + 4\cos^2\theta\,d\theta$$

$$= 200\int_0^{2\pi} 9 + 12\cos\theta + 2(1+\cos 2\theta)\,d\theta$$

$$= 200\int_0^{2\pi} 11 + 12\cos\theta + 2\cos 2\theta\,d\theta$$

$$= 200\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{2\pi}$$

$$= 200\left(\left(11\times 2\pi + 12\sin(2\pi) + \sin(2\times 2\pi)\right) - 0\right)$$

$$= 200(22\pi) = 4400\pi$$

$$= 13823.0\ \text{cm}^2$$

$$\text{Volume} = \text{Area}\times 90 = 396000\pi = \boxed{1244070.691}\ \text{cm}^3$$

Convert 50litres to cm^3

Since $1l = 0.001m^3$, $50l = 0.05m^3$

But $1m = 100cm$. So $1m^3 = 100^3 cm^3$

$\Rightarrow 50l = 0.05 \times 100^3 cm^3$

$\Rightarrow 50l = \boxed{50000\ cm^3}$

So time $= \dfrac{\boxed{1244070.691}}{\boxed{50000}} = 25$ minutes

Solution 6d

Model limitations (only one required):

- Polar equation unlikely to be accurate;
- Sides of pond will not be smooth so inaccurate conclusion will be drawn;
- Hole may not be of uniform depth;
- Pond may leak or absorb water.

7. **Solutions based entirely on graphical or numerical methods are not acceptable.**

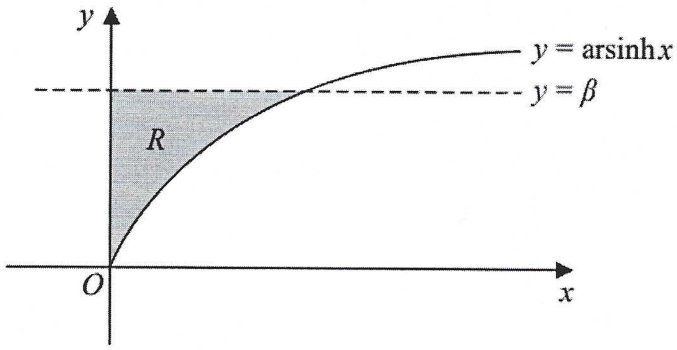


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = \operatorname{arsinh} x \qquad x \geqslant 0$$

and the straight line with equation $y = \beta$

The line and the curve intersect at the point with coordinates (α, β)

Given that $\beta = \frac{1}{2}\ln 3$

(a) show that $\alpha = \frac{1}{\sqrt{3}}$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the curve with equation $y = \operatorname{arsinh} x$, the y-axis and the line with equation $y = \beta$

The region R is rotated through 2π radians about the y-axis.

(b) Use calculus to find the exact value of the volume of the solid generated.

(6)

Solution 7a

Plug in co-ordinates (α, β) in equation

$$y = \operatorname{arsinh} x \quad x \geqslant 0$$

to give

$$\frac{1}{2}\ln 3 = \operatorname{arsinh} \alpha$$

But $\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$ (formula booklet)

$$\Rightarrow \quad \frac{1}{2}\ln 3 = \ln(\alpha + \sqrt{\alpha^2+1})$$

$$\Rightarrow \quad \ln 3^{1/2} = \ln(\alpha + \sqrt{\alpha^2+1})$$

$$\Rightarrow \quad 3^{1/2} = \alpha + \sqrt{\alpha^2+1} \quad \Rightarrow \quad (3^{1/2} - \alpha)^2 = \alpha^2 + 1$$

$$\Rightarrow \quad 3 - 2\sqrt{3}\alpha + \alpha^2 = \alpha^2 + 1 \quad \Rightarrow \quad \alpha = \frac{1}{\sqrt{3}}$$

Solution 7b

$$\text{Volume} = \pi \int_0^{\frac{1}{2}\ln 3} x^2\,dy$$

$$= \pi \int_0^{\frac{1}{2}\ln 3} \sinh^2 y\,dy$$

$$= \frac{\pi}{2}\int_0^{\frac{1}{2}\ln 3} \cosh 2x - 1\,dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2x - x\right]_0^{\frac{1}{2}\ln 3}$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\sinh \ln 3 - \frac{1}{2}\ln 3\right)$$

$$= \frac{\pi}{4}\left(\sinh \ln 3 - \ln 3\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}\left(e^{\ln 3} - e^{-\ln 3}\right) - \ln 3\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}\left(3 - e^{\ln 3^{-1}}\right) - \ln 3\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}\left(3 - 3^{-1}\right) - \ln 3\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}\left(3 - \frac{1}{3}\right) - \ln 3\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}\left(\frac{8}{3}\right) - \ln 3\right)$$

$$= \frac{\pi}{4}\left(\frac{4}{3} - \ln 3\right)$$

8. (i) The point P is one vertex of a regular pentagon in an Argand diagram. The centre of the pentagon is at the origin.

Given that P represents the complex number $6 + 6\mathrm{i}$, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$

(5)

(ii) (a) On a single Argand diagram, shade the region, R, that satisfies both

$$|z - 2\mathrm{i}| \leqslant 2 \quad \text{and} \quad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$

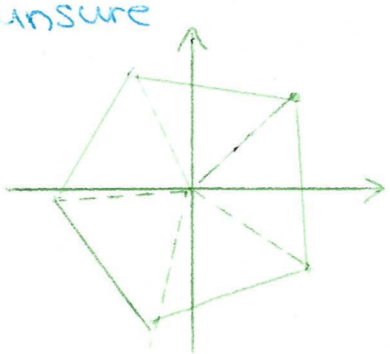
(2)

(b) Determine the exact area of R, giving your answer in simplest form.

(4)

Solution 8i

Try to visualise first if unsure



$z = 6 + 6i$

$\Rightarrow |z| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$

$\arg(z) = \tan^{-1}\left(\frac{6}{6}\right) = \tan^{-1} 1$

$= \frac{\pi}{4}$

$\Rightarrow z = 6\sqrt{2}e^{i\pi/4}$

To obtain the other vertices, add multiples of $\frac{2\pi}{5}$ to the argument of $z = 6\sqrt{2}e^{i\pi/4}$

So, vertices are

$z = 6\sqrt{2}e^{i\pi/4}$, $6\sqrt{2}e^{i\pi/4 + 2\pi/5}$, $6\sqrt{2}e^{i\pi/4 + 4\pi/5}$, $6\sqrt{2}e^{i\pi/4 + 6\pi/5}$

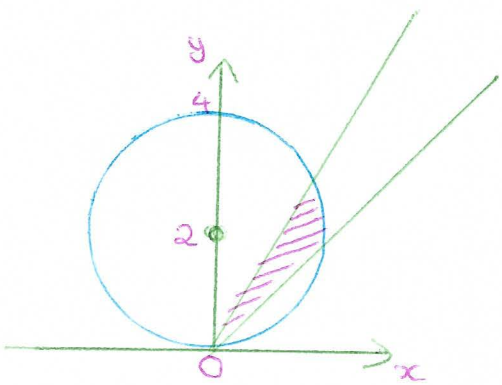
and $6\sqrt{2}e^{i\pi/4 + 8\pi/5}$

So $z = 6\sqrt{2}e^{i\pi/4}$, $6\sqrt{2}e^{13\pi/20}$, $6\sqrt{2}e^{21\pi/20}$, $6\sqrt{2}e^{29\pi/20}$, $6\sqrt{2}e^{37\pi/20}$

Solution 8iia

$|z-2i| \leq 2 \Rightarrow \sqrt{(x)^2+(y-2)^2} \leq 2$

$\Rightarrow$ Circle centre (0,2) radius ≤ 2



Solution 8iib

From formula booklet:
$\text{Area} = \frac{1}{2}\int r^2 \, d\theta$

How to obtain $4\sin\theta$ (in the integral):
Now $\sqrt{x^2+(y-2)^2} \leq 2$
$\Rightarrow x^2+y^2-4y+4 \leq 4$
$\Rightarrow (r\cos\theta)^2+(r\sin\theta)^2-4r\sin\theta \leq 0$
$\Rightarrow r^2(\cos^2\theta+\sin^2\theta) \leq 4r\sin\theta$
$\Rightarrow r^2 \leq 4r\sin\theta$
$\Rightarrow \boxed{r \leq 4\sin\theta}$

$$\text{Area} = \frac{1}{2}\int_{\pi/4}^{\pi/3} \boxed{(4\sin\theta)^2} \, d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/3} 16\boxed{\sin^2\theta} \, d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/3} 16\boxed{\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)} \, d\theta$$

$$= 4\int_{\pi/4}^{\pi/3} 1-\cos 2\theta \, d\theta$$

$$= 4\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$= 4\left(\frac{\pi}{3}-\frac{1}{2}\sin\frac{2\pi}{3}\right) - 4\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)$$

$$= \frac{4\pi}{3} - 2\sin\frac{2\pi}{3} - \pi + 2\sin\frac{\pi}{2}$$

$$= \frac{4\pi}{3} - 2\times\frac{\sqrt{3}}{2} - \pi + 2 = \frac{\pi}{3} - \sqrt{3} + 2$$

9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

(2)

Solution 9a

$$1+z+z^2+z^3+\ldots = \frac{1}{1-z}$$

Solution 9bi

$$1+z+z^2+z^3+\ldots = 1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{2^2}(\cos\theta+i\sin\theta)^2+\frac{1}{2^3}(\cos\theta+i\sin\theta)^3+\ldots$$

de Moivre

$$=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{2^2}(\cos 2\theta+i\sin 2\theta)+\frac{1}{2^3}(\cos 3\theta+i\sin 3\theta)+\ldots \quad (*)$$

But, by part a,

$$1+z+z^2+z^3+\ldots = \frac{1}{1-z} = \frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)}$$

$$= \frac{2}{(2-\cos\theta)-i\sin\theta} \times \frac{(2-\cos\theta)+i\sin\theta}{(2-\cos\theta)+i\sin\theta}$$ (to make denom real)

$$= \frac{2(2-\cos\theta)+2i\sin\theta}{(2-\cos\theta)^2+\sin^2\theta} = \frac{2(2-\cos\theta)+2i\sin\theta}{4-4\cos\theta+\underbrace{\cos^2\theta+\sin^2\theta}_{=1}}$$

$$= \frac{2(2-\cos\theta)+2i\sin\theta}{5-4\cos\theta} \quad (**)$$

Equating imaginary parts in (*) and (**) (since both expressions are equal to $1+z+z^2+z^3+\ldots$):

$$\frac{1}{2}\sin\theta+\frac{1}{2^2}\sin 2\theta+\frac{1}{2^3}\sin 3\theta+\ldots = \frac{2\sin\theta}{5-4\cos\theta}$$

Solution 9bii Try proof by contradiction;
if $1+z+z^2+\dots$ can be purely imaginary, then for some θ:

$$(*) \Rightarrow \frac{2(2-\cos\theta)}{5-4\cos\theta} = 0$$

$$\Rightarrow \cos\theta = 2$$

Clearly this is a contradiction since

$$-1 \leqslant \cos\theta \leqslant 1$$

Hence $1+z+z^2+\dots$ cannot be purely imaginary.