1. A particle P moves with constant acceleration $(2\mathbf{i} - 3\mathbf{j})\,\mathrm{m\,s^{-2}}$

At time $t = 0$, P is moving with velocity $4\mathbf{i}\,\mathrm{m\,s^{-1}}$

(a) Find the velocity of P at time $t = 2$ seconds.

(2)

At time $t = 0$, the position vector of P relative to a fixed origin O is $(\mathbf{i} + \mathbf{j})\,\mathrm{m}$.

(b) Find the position vector of P relative to O at time $t = 3$ seconds.

(2)

Solution 1a

As acceleration is constant;

$$\underline{v} = \boxed{\underline{u}} + \boxed{\underline{a}}\boxed{t} \qquad \underline{u} = \boxed{\begin{pmatrix} 4 \\ 0 \end{pmatrix}}, \quad \boxed{t = 2}$$

$$\underline{v} = \boxed{\begin{pmatrix} 4 \\ 0 \end{pmatrix}} + \boxed{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}\boxed{(2)}$$

$$= \begin{pmatrix} 4 + 4 \\ 0 - 6 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$$

$$\Rightarrow \underline{v} = 8\underline{i} - 6\underline{j}$$

Solution 1b

$$\underline{r} = \boxed{\underline{u}}t + \tfrac{1}{2}\boxed{\underline{a}}t^2, \quad \boxed{t = 3}$$

$$\text{Position: } \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} + \left[\boxed{\begin{pmatrix} 4 \\ 0 \end{pmatrix}}\boxed{(3)} + \tfrac{1}{2}\boxed{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}\boxed{(3)^2}\right]$$

position at time $t=0$

$$= \begin{pmatrix} 1 + 12 + 9 \\ 1 + 0 - 27/2 \end{pmatrix}$$

$$= \begin{pmatrix} 22 \\ -12.5 \end{pmatrix} = 22\underline{i} - 12.5\underline{j}$$

2.

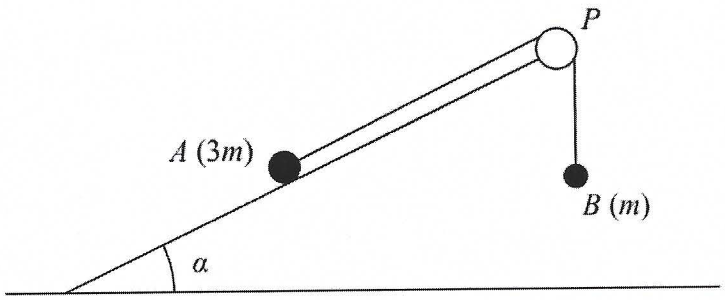


Figure 1

A small stone A of mass $3m$ is attached to one end of a string.

A small stone B of mass m is attached to the other end of the string.

Initially A is held at rest on a fixed rough plane.

The plane is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{3}{4}$

The string passes over a pulley P that is fixed at the top of the plane.

The part of the string from A to P is parallel to a line of greatest slope of the plane.

Stone B hangs freely below P, as shown in Figure 1.

The coefficient of friction between A and the plane is $\frac{1}{6}$

The stones are modelled as particles.

The pulley is modelled as being small and smooth.

The string is modelled as being light and inextensible.

Using the model for the motion of the system before B reaches the pulley,

(a) write down an equation of motion for A

(2)

(b) show that the acceleration of A is $\frac{1}{10}g$

(7)

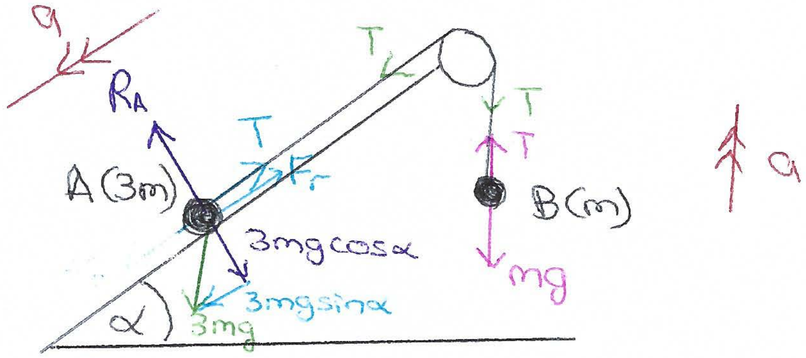
(c) sketch a velocity-time graph for the motion of B, from the instant when A is released from rest to the instant just before B reaches the pulley, explaining your answer.

(2)

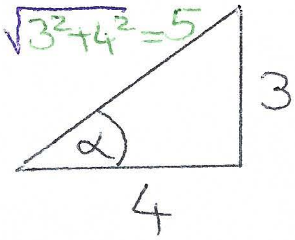
In reality, the string is not light.

(d) State how this would affect the working in part (b).

(1)



Solution 2a



$R_A(\nearrow)$: $\underline{F}=m\underline{a}$

$3mg\sin\alpha - F_r - T = 3ma$

$\Rightarrow 3mg\left(\frac{3}{5}\right) - F_r - T = 3ma$

$\Rightarrow \frac{9}{5}mg - F_r - T = 3ma \quad (*)$

$\tan\alpha = \frac{3}{4}$

$\Rightarrow \cos\alpha = \frac{4}{5}$

$\Rightarrow \sin\alpha = \frac{3}{5}$

Solution 2b

$R_A(\nwarrow)$:

$R = 3mg\cos\alpha$

$\Rightarrow R = 3mg\left(\frac{4}{5}\right)$

$\Rightarrow R = \frac{12mg}{5}$

$R_B(\uparrow)$: $\underline{F}=m\underline{a}$

$T - mg = ma \quad (**)$

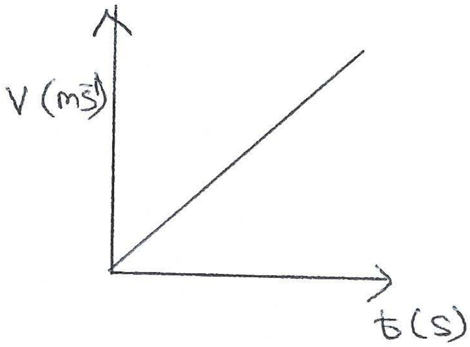
$(*) + (**) \Rightarrow \frac{9}{5}mg - F_r - T + T - mg = 3ma + ma$

$\Rightarrow \frac{9}{5}mg - F_r - mg = 4ma$

$\Rightarrow \frac{4}{5}mg - \frac{1}{6}R = 4ma$

$\Rightarrow \frac{4}{5}mg - \frac{1}{6}\left(\frac{12}{5}\right)mg = 4ma \Rightarrow a = \frac{1}{10}g$

Solution 2c



Acceleration is constant
So gradient of graph
is constant.

Solution 2d

The tension in the string would not be the same everywhere, so the tension (T) on A would be different to the tension on B.

3.

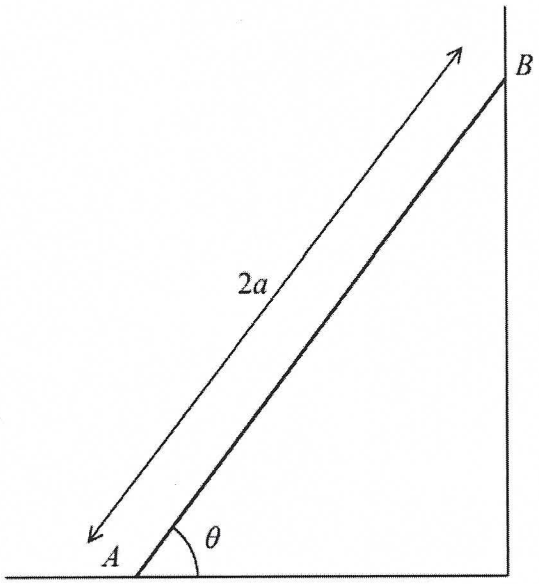


Figure 2

A beam AB has mass m and length $2a$.

The beam rests in equilibrium with A on rough horizontal ground and with B against a smooth vertical wall.

The beam is inclined to the horizontal at an angle θ, as shown in Figure 2.

The coefficient of friction between the beam and the ground is μ

The beam is modelled as a uniform rod resting in a vertical plane that is perpendicular to the wall.

Using the model,

(a) show that $\mu \geqslant \frac{1}{2}\cot\theta$

(5)

A horizontal force of magnitude kmg, where k is a constant, is now applied to the beam at A.

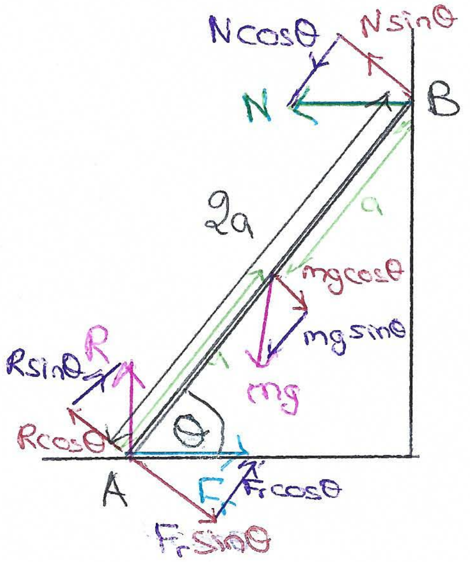
This force acts in a direction that is perpendicular to the wall and towards the wall.

Given that $\tan\theta = \frac{5}{4}$, $\mu = \frac{1}{2}$ and the beam is now in limiting equilibrium,

(b) use the model to find the value of k.

(5)

Solution 3a



$M(\overset{\curvearrowright}{A}) = mga\cos\theta$

$M(\overset{\curvearrowleft}{A}) = 2Na\sin\theta$

As the beam is in equilibrium:

$mga\cos\theta = 2Na\sin\theta$

$M(\overset{\curvearrowright}{B}) = 2aR\cos\theta$

$M(\overset{\curvearrowleft}{B}) = mga\cos\theta + 2F_r a\sin\theta$

As the beam is in equilibrium:

$2a\boxed{R}\cos\theta = mga\cos\theta + 2F_r a\sin\theta$

As $\boxed{R = mg}$ and $F_r = N$ (due to equilibrium)

Then $Ra\cos\theta = 2F_r a\sin\theta$

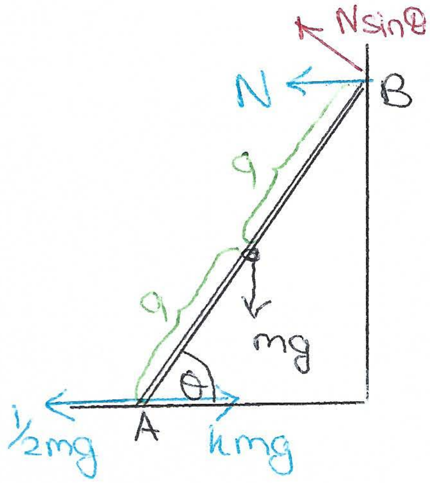
$$\Rightarrow \boxed{F_r} = \frac{Ra\cos\theta}{2a\sin\theta} = \boxed{\frac{1}{2}R\cot\theta}$$

As in equilibrium,

$\boxed{F_r} \leq \mu R \Rightarrow \frac{1}{2}R\cot\theta \leq \mu R$

$\Rightarrow \mu \geq \frac{1}{2}\cot\theta$

Solution 3b



$\theta = \tan^{-1}\left(\frac{5}{4}\right)$, $\mu = \frac{1}{2}$

$M(\overset{\curvearrowright}{A}) = mga\cos\theta$

$M(\overset{\curvearrowleft}{A}) = 2Na\sin\theta$

$mga\cos\theta = 2Na\sin\theta$

$N = kmg - F$ (due to equilibrium)

$F = \mu mg$ (due to limiting equilibrium)

$\Rightarrow mga\cos\theta = 2a(kmg - \mu mg)\sin\theta$

$\Rightarrow mga\cos\theta = 2mga(k-\mu)\sin\theta$

$\Rightarrow \quad \cos\theta = 2(k-\mu)\sin\theta$

$\Rightarrow \quad 1 = 2(k-\mu)\dfrac{\sin\theta}{\cos\theta}$

$\Rightarrow \quad 1 = \frac{5}{2}k - \frac{5}{2} \times \frac{1}{2}$

$\Rightarrow \quad k = 0.9$

4.

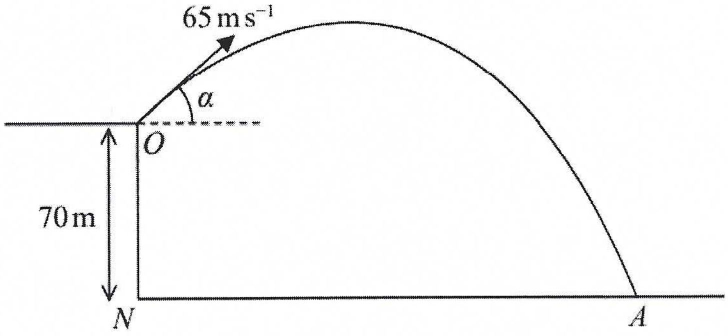


Figure 3

A small stone is projected with speed 65 m s^{-1} from a point O at the top of a vertical cliff.

Point O is 70 m vertically above the point N.

Point N is on horizontal ground.

The stone is projected at an angle α above the horizontal, where $\tan\alpha = \dfrac{5}{12}$

The stone hits the ground at the point A, as shown in Figure 3.

The stone is modelled as a particle moving freely under gravity.

The acceleration due to gravity is modelled as having magnitude 10 m s^{-2}

Using the model,

(a) find the time taken for the stone to travel from O to A,

(4)

(b) find the speed of the stone at the instant just before it hits the ground at A.

(5)

One limitation of the model is that it ignores air resistance.

(c) State one other limitation of the model that could affect the reliability of your answers.

(1)

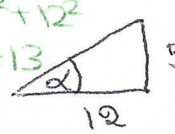
Solution 4a

(Resolving vertically) $s = ut + \frac{1}{2}at^2$

$-70 = (65\sin\alpha)t + \frac{1}{2}(-10)t^2 \quad (a = -10)$

$\Rightarrow -70 = 25t - 5t^2$ ← Substituting $\sin\alpha = \frac{5}{13}$ and rearranging

$\sqrt{5^2+12^2} = 13$



$\tan\alpha = \frac{5}{12} \Rightarrow \sin\alpha = \frac{5}{13}$

$\Rightarrow (t-7)(t+2) = 0$

As $t \neq -2$ we have $t = 7$.

Solution 4b

→ Horizontal component of velocity is modelled as constant:

$$V_h = 65\cos\alpha = 65\left(\frac{12}{13}\right) = \boxed{60}\,ms^{-1}$$

(at start)

↑ Vertical component of velocity:

$$V_v = \sqrt{(-65\sin\alpha)^2 + 2\times10\times70} \qquad v^2 = u^2 + 2as$$

$$\Rightarrow V_v = \sqrt{25^2 + 1400} = \boxed{45}\,ms^{-1}$$

Speed will then be the magnitude of these velocities

$$\text{Speed} = \sqrt{\boxed{60^2} + \boxed{45^2}} = 75\,ms^{-1}$$

Solution 4c

One limitation of the model is modelling the stone as a particle

5. At time t seconds, a particle P has velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$, where

$$\mathbf{v} = 3t^{\frac{1}{2}}\mathbf{i} - 2t\mathbf{j} \qquad t > 0$$

(a) Find the acceleration of P at time t seconds, where $t > 0$

(2)

(b) Find the value of t at the instant when P is moving in the direction of $\mathbf{i} - \mathbf{j}$

(3)

At time t seconds, where $t > 0$, the position vector of P, relative to a fixed origin O, is $\mathbf{r}$ metres.

When $t = 1$, $\mathbf{r} = -\mathbf{j}$

(c) Find an expression for $\mathbf{r}$ in terms of t.

(3)

(d) Find the exact distance of P from O at the instant when P is moving with speed $10\,\mathrm{m\,s^{-1}}$

(6)

Solution 5a

$\underline{v} = \begin{pmatrix} 3t^{1/2} \\ -2t \end{pmatrix}$ Now $\underline{a} = \frac{d\underline{v}}{dt} \Rightarrow \underline{a} = \begin{pmatrix} \frac{3}{2}t^{-1/2} \\ -2 \end{pmatrix} \mathrm{ms^{-2}}$

Solution 5b

P is travelling in direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ when

$$\begin{pmatrix} 3t^{1/2} \\ -2t \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}$$

$$\Rightarrow -2t = -\lambda \Rightarrow \lambda = 2t$$

Also, $3t^{1/2} = \lambda = 2t$

$$\Rightarrow 3t^{1/2} = 2t$$
$$\Rightarrow 9t = 4t^2 \Rightarrow t(4t-9) = 0 \Rightarrow t = 9/4 \text{ seconds}$$

Solution 5c

$\underline{r} = \int \underline{v}\, dt = \int \begin{pmatrix} 3t^{1/2} \\ -2t \end{pmatrix} dt = \begin{pmatrix} 2t^{3/2} \\ -t^2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

At $t = 1$, $\underline{r} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

So $\underline{r} = \begin{pmatrix} 2t^{3/2} - 2 \\ -t^2 \end{pmatrix}$